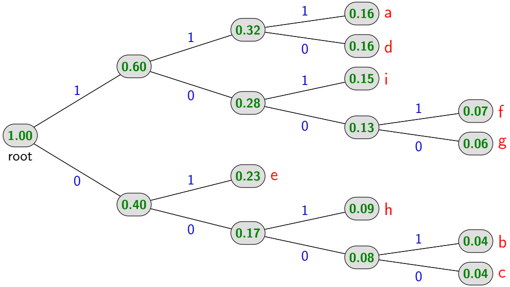
# Optimal Variable-Length Codes

| $a_k$ | $p_k$ | codewords |
|---|---|---|
| a | 0.16 | 111 |
| b | 0.04 | 0001 |
| c | 0.04 | 0000 |
| d | 0.16 | 110 |
| e | 0.23 | 01 |
| f | 0.07 | 1001 |
| g | 0.06 | 1000 |
| h | 0.09 | 001 |
| i | 0.15 | 101 |



# Last Lecture: Scalar Variable-Length Codes

## Unique Decodability

- Necessary condition: Kraft-McMillan inequality for codeword lengths: $\sum_k 2^{-\ell_k} \leq 1$
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- Entropy = lower bound for $\bar{\ell}$: $\bar{\ell} \geq H(p)$
- Can always construct prefix code with $H(p) \leq \bar{\ell} < H(p) + 1$ (e.g., Shannon code)

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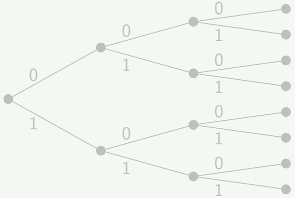

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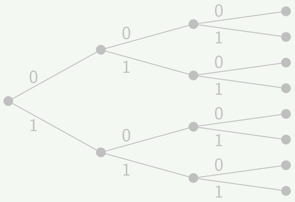

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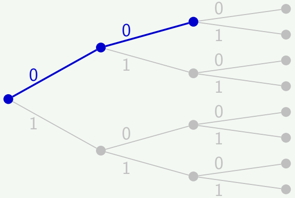

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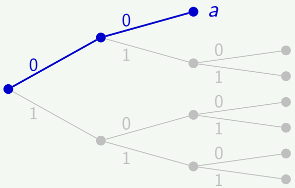

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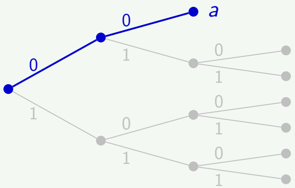

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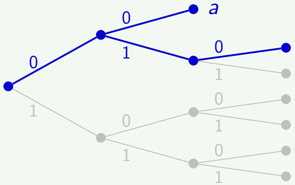

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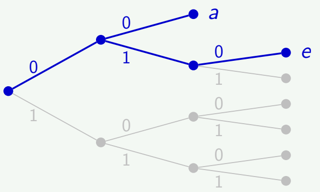

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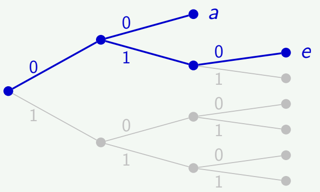

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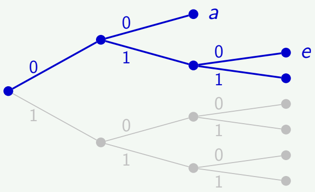

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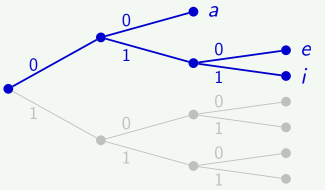

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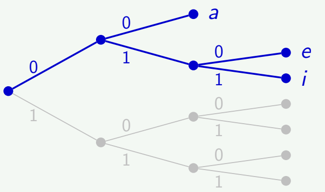

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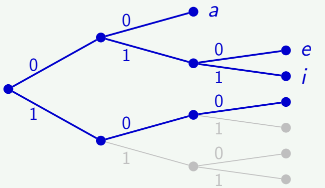

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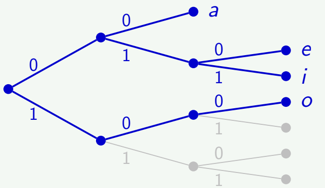

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$$\bar{\ell} \geq H(p)$$

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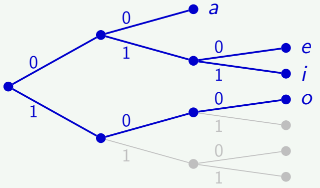

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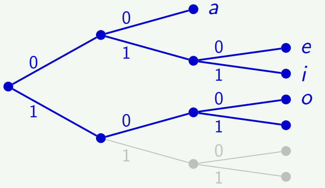

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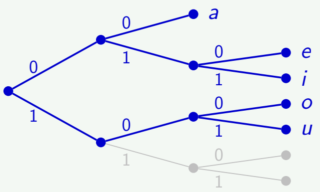

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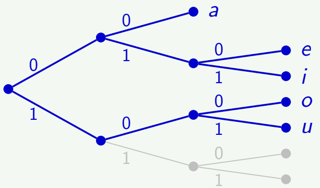

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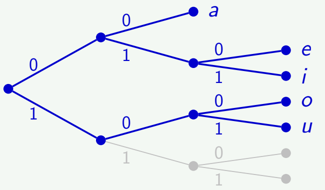


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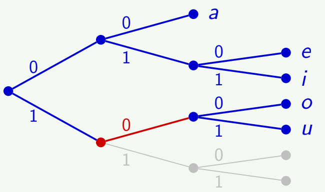


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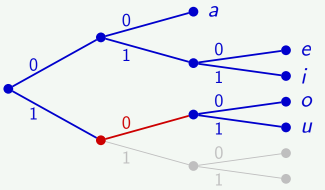


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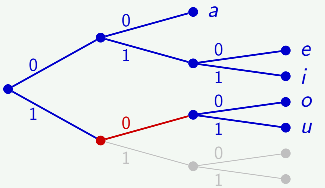


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Note: Removing the redundant bit would yield $\bar{\ell} = 2.3$, $\varrho \approx 5.9\%$ (still not optimal)

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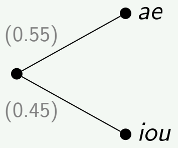

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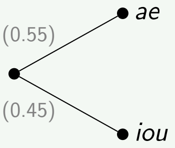

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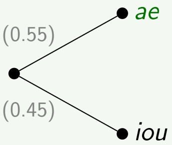

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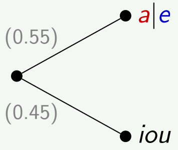

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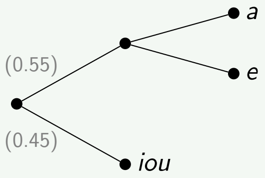

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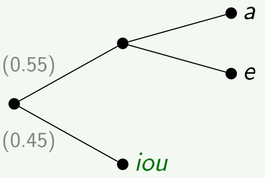

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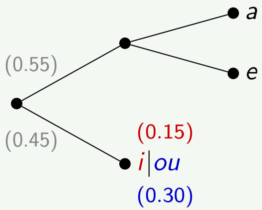

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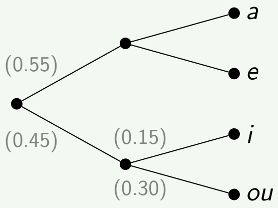

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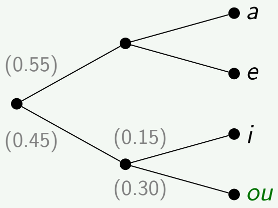

# Shannon-Fano Code: Construct Full Binary Code Tree

1. Sort symbols in alphabet according to their probability masses
2. Divide sorted list into two groups $A$ and $B$, so that $P(A)$ is as close to $P(B)$ as possible
3. Create a node and assign the first group $A$ to one branch and the other group $B$ to the other branch
4. Recursively apply steps 2 and 3 to the groups $A$ and $B$ until all symbols are assigned to terminal nodes

## Example: Shannon-Fano Code

| $a_k$ | $p_k$ |
|---|---|
| $a$ | 0.40 |
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| $i$ | 0.15 |
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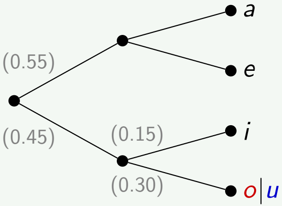

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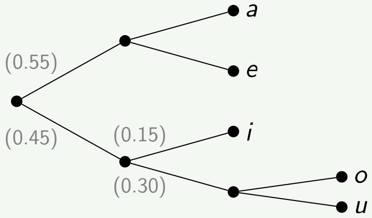

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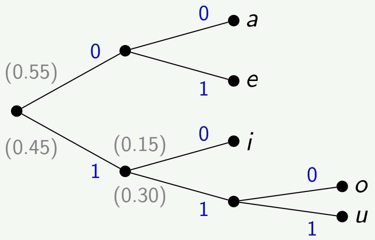

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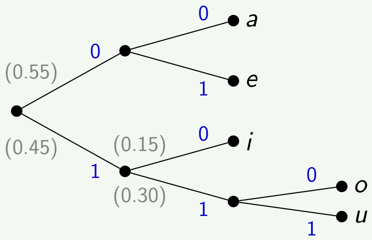
## Example: Shannon-Fano Code

| $a_k$ | $p_k$ | codewords |
|---|---|---|
| $a$ | 0.40 | 00 |
| $e$ | 0.15 | 01 |
| $i$ | 0.15 | 10 |
| $o$ | 0.15 | 110 |
| $u$ | 0.15 | 111 |
| $H \approx 2.171$ | | $\bar{\ell} = 2.3$<br>$\varrho \approx 5.9\%$ |



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## Example: Shannon-Fano Code

| $a_k$ | $p_k$ | codewords | better code |
|---|---|---|---|
| $a$ | 0.40 | 00 | 0 |
| $e$ | 0.15 | 01 | 100 |
| $i$ | 0.15 | 10 | 101 |
| $o$ | 0.15 | 110 | 110 |
| $u$ | 0.15 | 111 | 111 |
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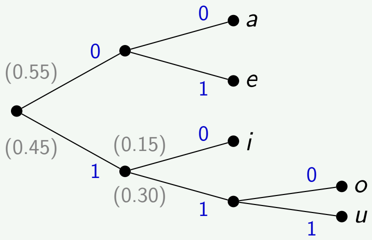

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➞ **No Guarantee for Optimality** **(but yields prefix code without structural redundancy)**

## Example: Shannon-Fano Code

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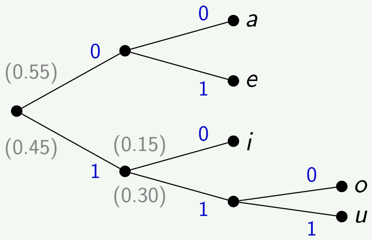

# Example: Shannon Code / Shannon-Fano Code / Optimal Code

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| $a_k$ | $p_k$ | Shannon |
|---|---|---|
| $a$ | 0.40 | 00 |
| $e$ | 0.15 | 010 |
| $i$ | 0.15 | 011 |
| $o$ | 0.15 | 100 |
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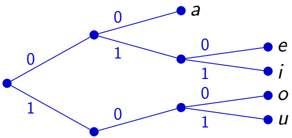
| $H \approx 2.171$ | | $\bar{\ell} = 2.6$<br>$\varrho \approx 19.8\%$ |

**Shannon code**

$\ell_k = \lceil -\log_2 p_k \rceil$



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| $a$ | 0.40 | 00 | 00 |
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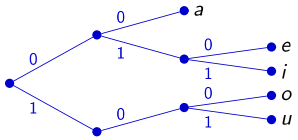
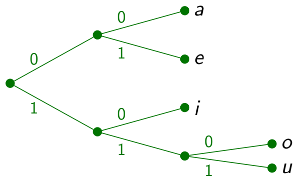



**Shannon-Fano code**

recursive equal probability split



# Example: Shannon Code / Shannon-Fano Code / Optimal Code

| $a_k$ | $p_k$ | Shannon | Shannon-Fano | optimal |
|---|---|---|---|---|
| $a$ | 0.40 | 00 | 00 | 0 |
| $e$ | 0.15 | 010 | 01 | 100 |
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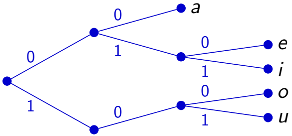


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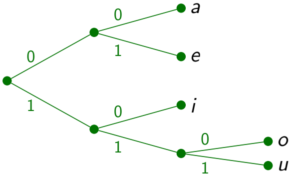
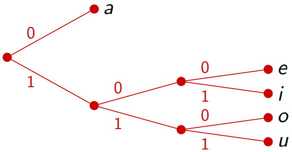



**optimal code**

? test all full binary trees with given number of terminal nodes



# Example: Shannon Code / Shannon-Fano Code / Optimal Code

| $a_k$ | $p_k$ | Shannon | Shannon-Fano | optimal |
|---|---|---|---|---|
| $a$ | 0.40 | 00 | 00 | 0 |
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**Question**

**Is there a low-complex algorithm for constructing optimal prefix codes ?**

## Shannon code

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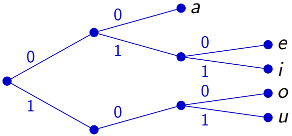


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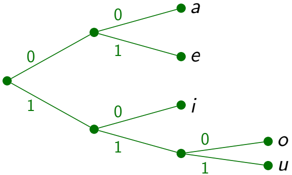


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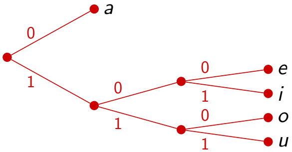

# Optimal Lossless Codes

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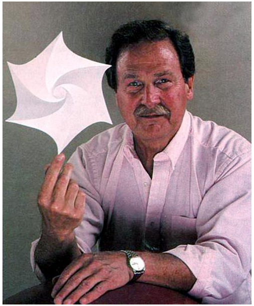
## Construction of Optimal Prefix Codes ?

➡ Finite alphabets: **Huffman algorithm** (1952) yields a prefix code with minimum redundancy



1098 *PROCEEDINGS OF THE I.R.E.* *September*

### A Method for the Construction of Minimum-Redundancy Codes*

DAVID A. HUFFMAN†, ASSOCIATE, IRE

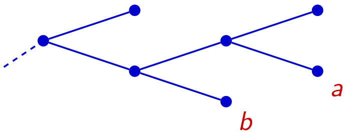
***Summary*—An optimum method of coding an ensemble of messages consisting of a finite number of members is developed. A minimum-redundancy code is one constructed in such a way that the average number of coding digits per message is minimized.**

INTRODUCTION

ONE IMPORTANT METHOD of transmitting messages is to transmit in their place sequences of symbols. If there are more messages which will be defined here as an ensemble code which, for a message ensemble consisting of a finite number of members, $N$, and for a given number of coding digits, $D$, yields the lowest possible average message length. In order to avoid the use of the lengthy term "minimum-redundancy," this term will be replaced here by "optimum." It will be understood then that, in this paper, "optimum code" means "minimum-redundancy code."

# Exchanging Codewords in a Prefix Code

- Consider any prefix code for an alphabet $\mathcal{A}$ which includes two letters $a$ and $b$ with associated probabilities $p_a$ and $p_b$



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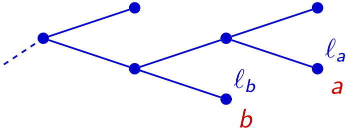

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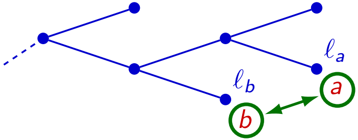


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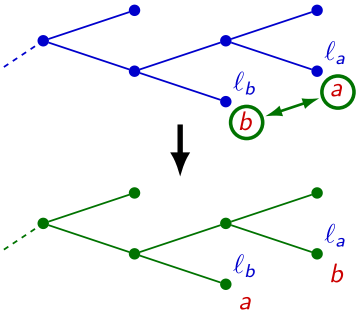


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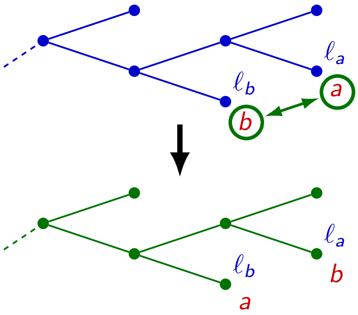

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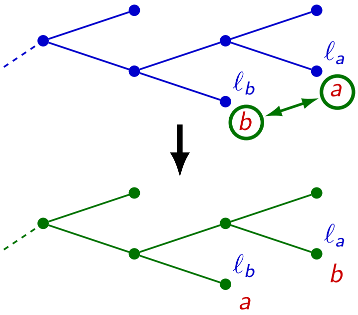


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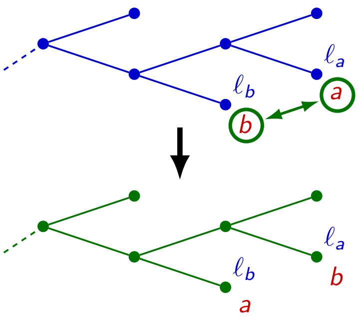


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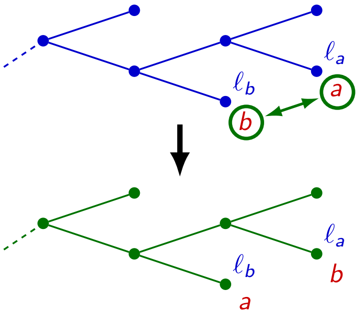


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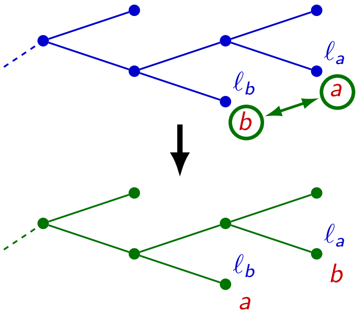


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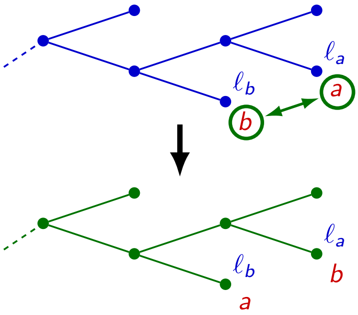


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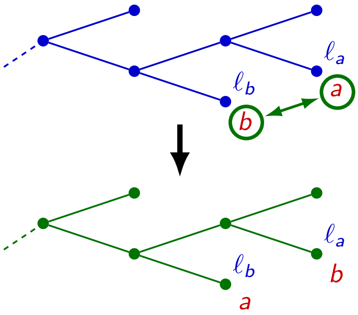


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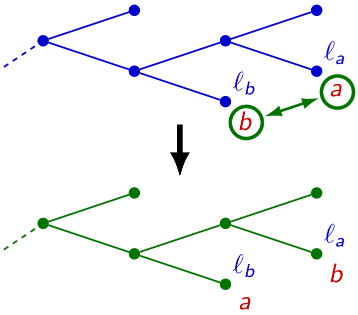
**(1)** $p_a = p_b$ or $\ell_a = \ell_b$: ➡ $\bar{\ell}_{\text{new}} = \bar{\ell}$

**(2)** $p_a > p_b$ and $\ell_a < \ell_b$: ➡ $\bar{\ell}_{\text{new}} > \bar{\ell}$

# Exchanging Codewords in a Prefix Code

- Consider any prefix code for an alphabet $\mathcal{A}$ which includes two letters $a$ and $b$ with associated probabilities $p_a$ and $p_b$
- $a$ and $b$ are associated with codewords of lengths $\ell_a$ and $\ell_b$




**What happens if we exchange the codewords for $a$ and $b$?**

➡ Obtain a new prefix code

➡ Average codeword length changes from $\bar{\ell}$ to $\bar{\ell}_{\text{new}}$

$$\begin{aligned} \bar{\ell}_{\text{new}} &= \bar{\ell} - (p_a\ell_a + p_b\ell_b) + (p_a\ell_b + p_b\ell_a) \\ &= \bar{\ell} - p_a(\ell_a - \ell_b) + p_b(\ell_a - \ell_b) \\ &= \bar{\ell} - (p_a - p_b)(\ell_a - \ell_b) \end{aligned}$$

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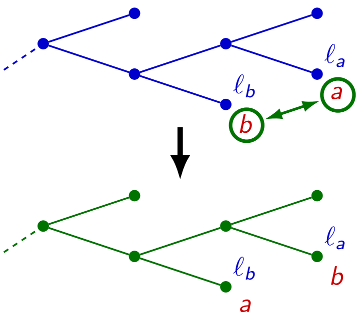


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**In an optimal prefix code, we require**

$$\forall a, b: \quad p_a > p_b \implies \ell_a \leq \ell_b$$

# Subset of Optimal Prefix Codes

**Lemma (class of optimal prefix codes)**

For any finite alphabet $\mathcal{A}$, there exists an optimal prefix code $\mathcal{C}$ with the following property:

> There are two codewords that have the maximum length, differ only in the final bit, and correspond to the two least likely alphabet letters.

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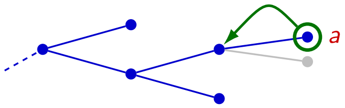
In the binary tree representation, this means that the corresponding terminal node has a sibling.

Codes without that property cannot be optimal, since a removal of the last bit of the considered codeword of maximum length would reduce the average codeword length without violating the prefix property.




$$\begin{aligned} \bar{\ell}_{\text{new}} &= \bar{\ell} - p_a \cdot \ell_a + p_a \cdot (\ell_a - 1) \\ &= \bar{\ell} - p_a \\ &< \bar{\ell} \end{aligned}$$

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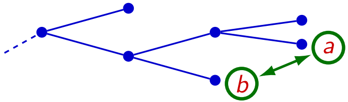


$$\bar{\ell}_{\text{new}} = \bar{\ell} - (p_a - p_b)(\ell_a - \ell_b)$$

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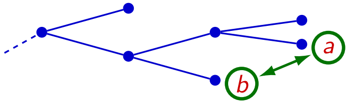
Note: For two alphabet letters $a$ and $b$ with $p_a = p_b$, an exchange of the codewords does not modify the average codeword length.

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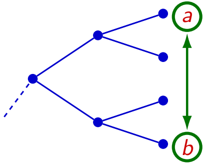
## Proof

**3** Two codewords of maximum length that differ only in the last bit are assigned to the two least likely alphabet letters.

Not necessary for optimality.

But we can always exchange any two codewords of maximum length (i.e., the same length $\ell_{\max}$) without impacting the average codeword length.




$$\bar{\ell}_{\text{new}} = \bar{\ell} - (p_a - p_b)(\ell_{\max} - \ell_{\max}) = \bar{\ell}$$

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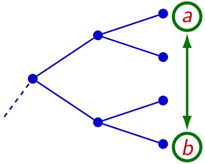


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➡ **There exists on optimal prefix code $\mathcal{C}$ with the above stated property.**

# The Huffman Algorithm

## Idea for Construction of Binary Code Tree

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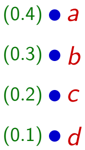

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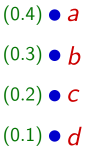

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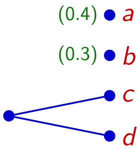

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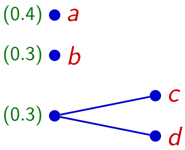

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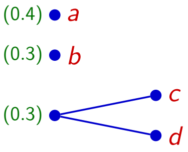

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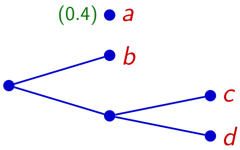

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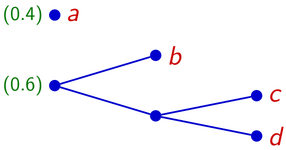

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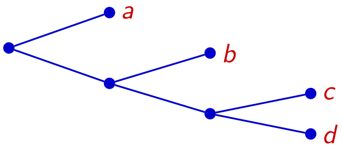

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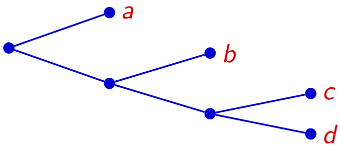


### Huffman Algorithm (via construction of binary code tree)

1. Select the two letters $a$ and $b$ with the smallest probabilities $p_a$ and $p_b$
2. Create a parent node for the two letters $a$ and $b$ in the binary code tree
3. Replace the letters $a$ and $b$ with a new letter with probability $p_a + p_b$
4. If the resulting new alphabet contains more than a single letter, repeat all previous steps with this alphabet
5. Convert the obtained binary code tree into a prefix code

# Example: Construction of a Huffman Code

given:
alphabet $\mathcal{A}$ with pmf $\{p_k\}$

| $a_k$ | $p_k$ | codewords |
|---|---|---|
| a | 0.16 | |
| b | 0.04 | |
| c | 0.04 | |
| d | 0.16 | |
| e | 0.23 | |
| f | 0.07 | |
| g | 0.06 | |
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first step:

assign symbols and probabilities to terminal nodes

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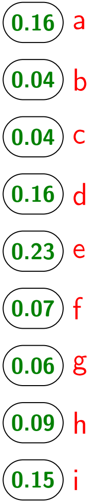

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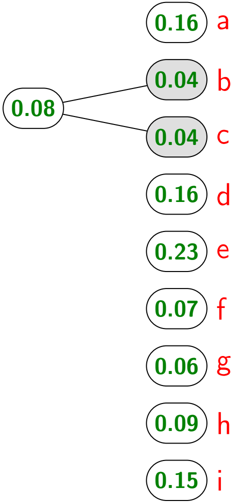

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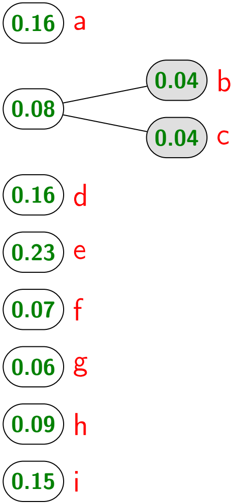

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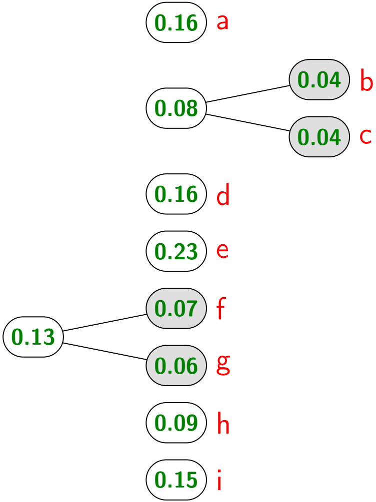

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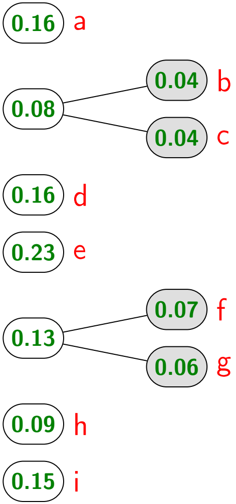

# Example: Construction of a Huffman Code

next step:

re-order for better readability

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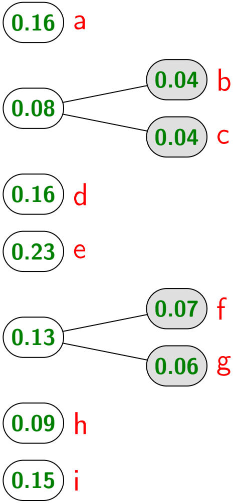

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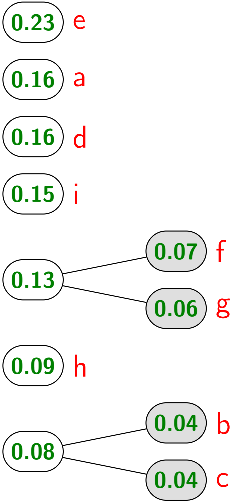

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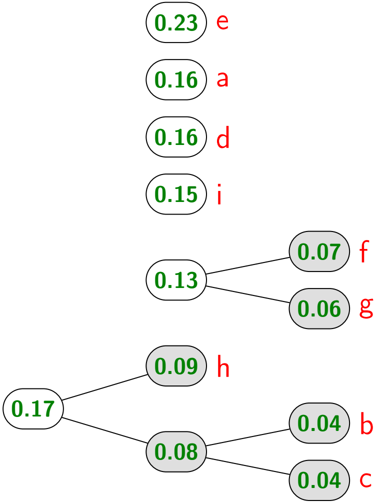

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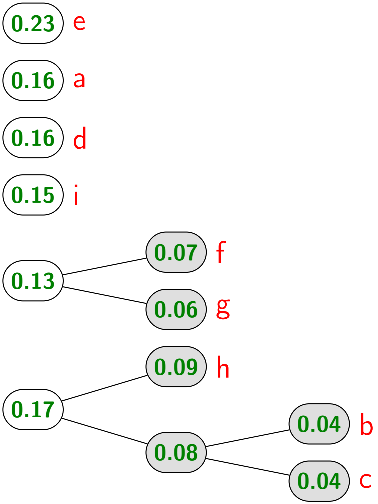

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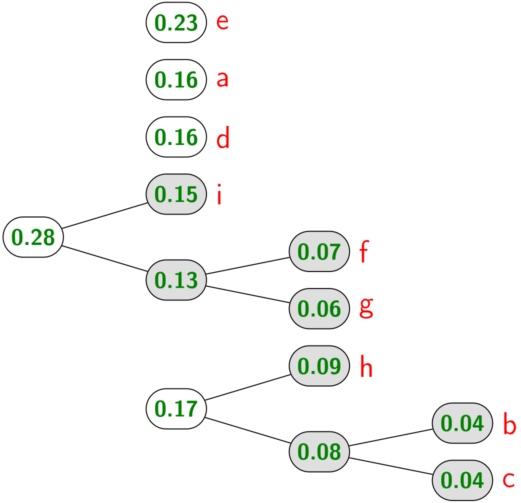

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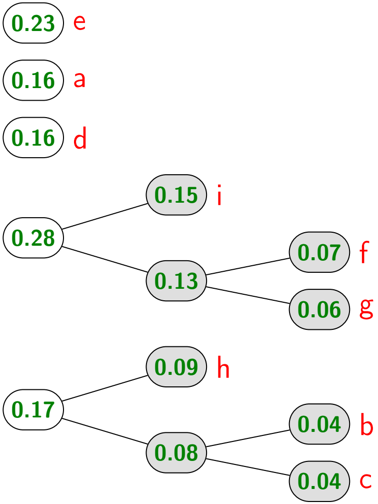

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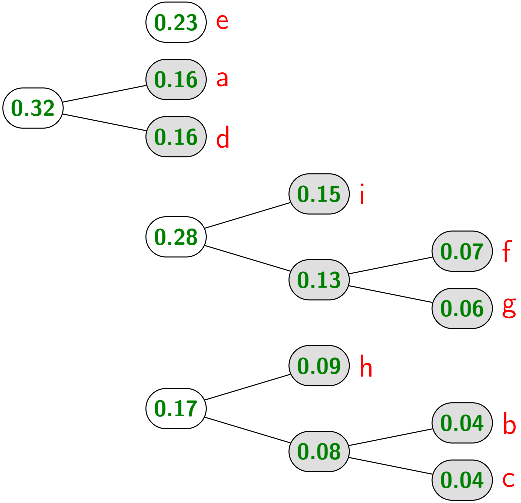

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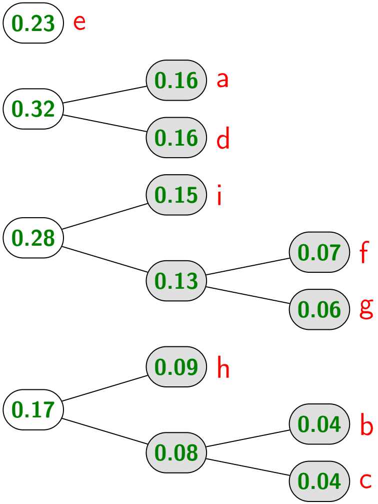

# Example: Construction of a Huffman Code

next step:

re-order for better readability

| $a_k$ | $p_k$ | codewords |
|---|---|---|
| a | 0.16 | |
| b | 0.04 | |
| c | 0.04 | |
| d | 0.16 | |
| e | 0.23 | |
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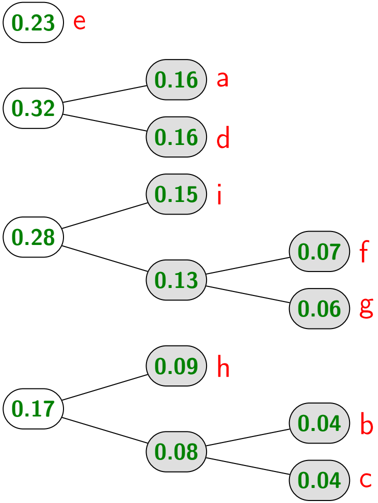

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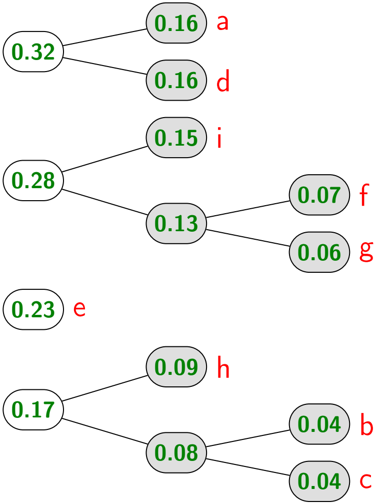

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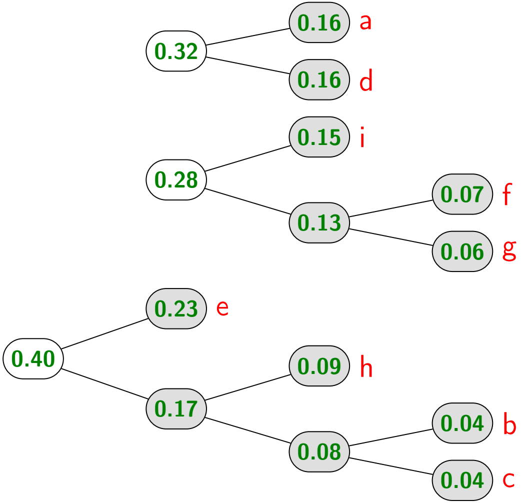

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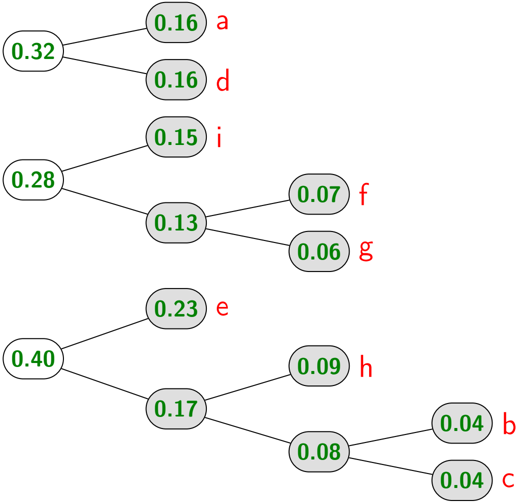

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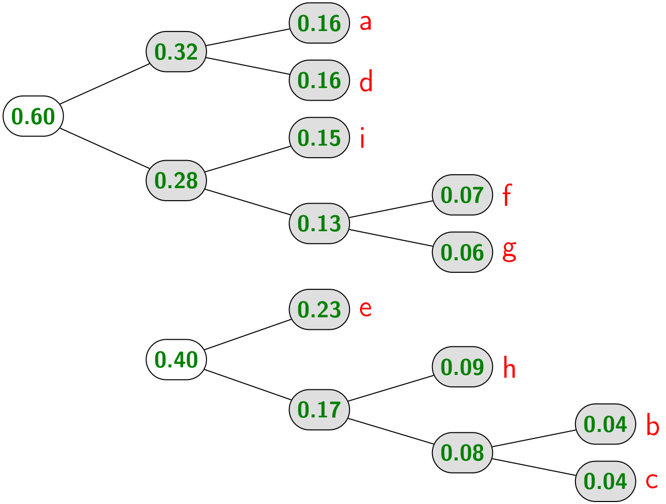

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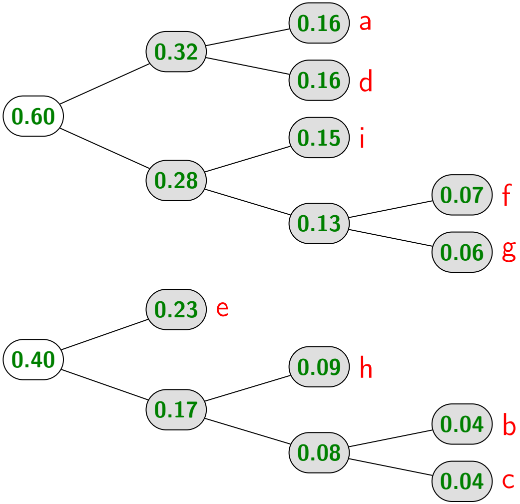

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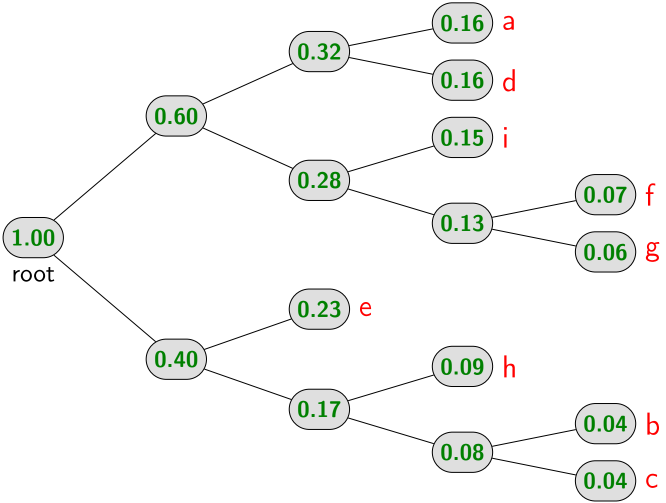

# Example: Construction of a Huffman Code

next step:

label branches with 0 and 1

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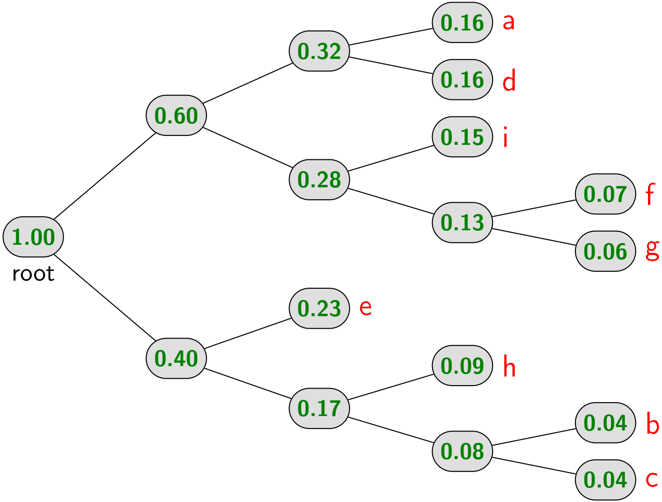

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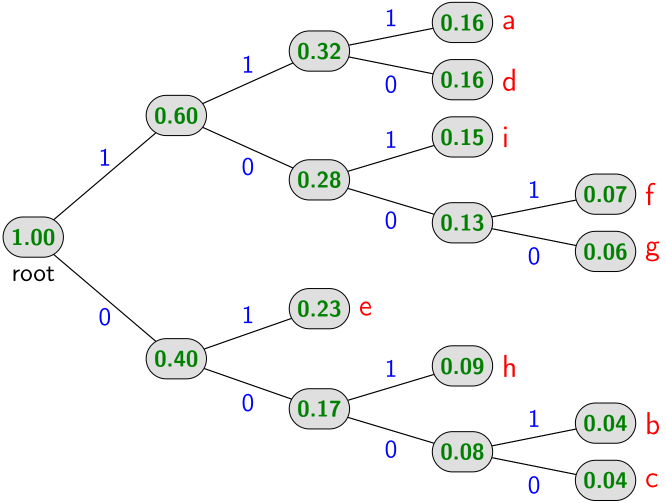

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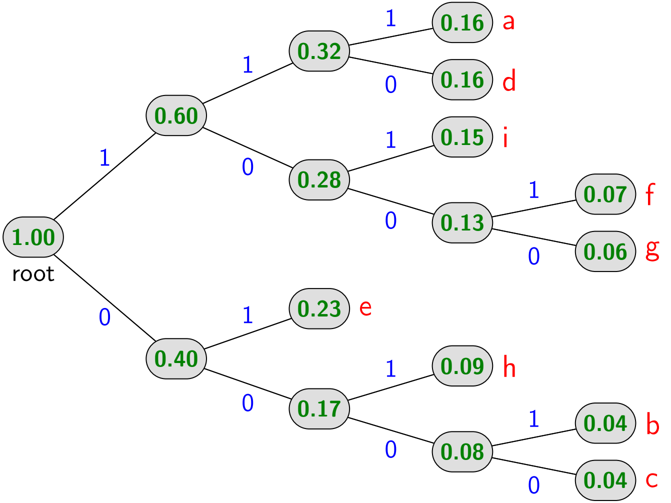

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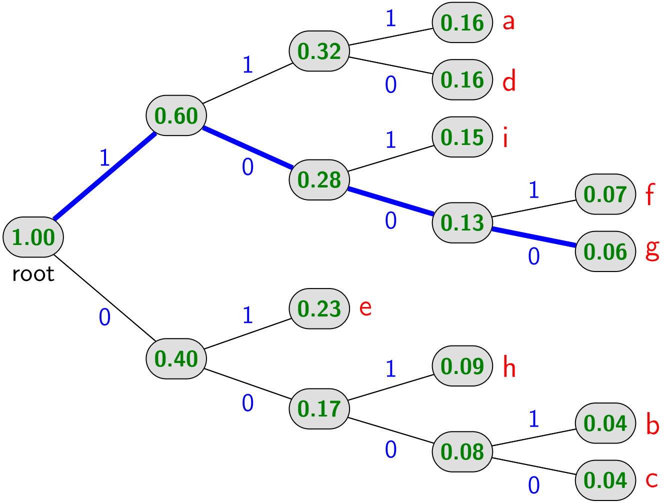
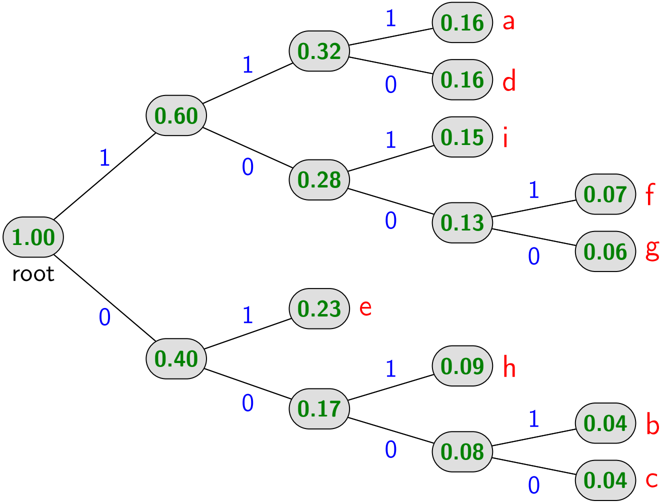
| g | 0.06 | 1000 |
| h | 0.09 | |
| i | 0.15 | |



# Example: Construction of a Huffman Code

| $a_k$ | $p_k$ | codewords |
|---|---|---|
| a | 0.16 | 111 |
| b | 0.04 | 0001 |
| c | 0.04 | 0000 |
| d | 0.16 | 110 |
| e | 0.23 | 01 |
| f | 0.07 | 1001 |
| g | 0.06 | 1000 |
| h | 0.09 | 001 |
| i | 0.15 | 101 |



# Example: Construction of a Huffman Code

$$\bar{\ell} = 2.98$$
$$H(p) \approx 2.9405$$
$$\varrho \approx 0.0395 \;(1.34\,\%)$$

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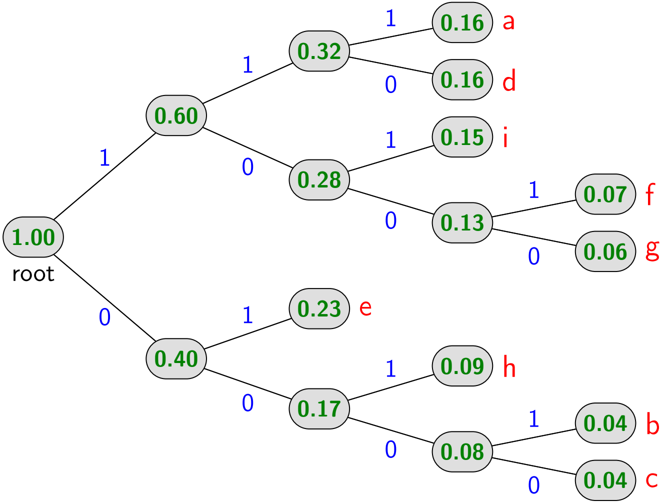

# Average Codeword Length of Huffman Codes

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**➡ Average codeword length of Huffman codes (optimal codes) is bounded by**

$$H(p) \leq \bar{\ell}_{\text{opt}} < H(p) + 1$$

# Unix File Compression Utility "Pack"

## Basic principle

- Design Huffman code for bytes of file to be compressed
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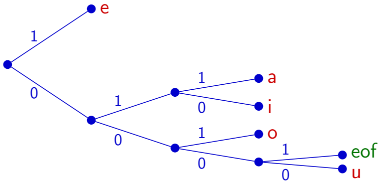

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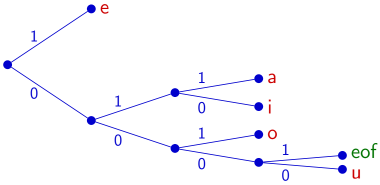


**syntax of bitstream**

| | |
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| 16 bits | magic header ( "1f 1e" ) |
| 32 bits | number of symbols |
| 8 bits | depth of Huffman tree |
| 8 bits | number of terminal nodes in level 1 |
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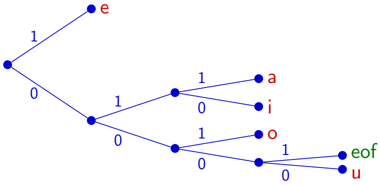


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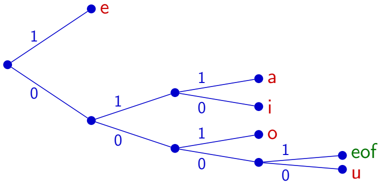
## Decoding

- Decode "Huffman tree" from the compressed bitstream
- Read variable-length codewords and output bytes

using one-sided Huffman trees (with eof)

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$$\bar{\ell} \approx 1.3556 \quad (\,^{61}/_{45}\,)$$
$$\varrho \approx 0.0980 \quad (\,7.8\,\%\,)$$

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| $c$ | $5/45$ | 00 |

$$H(S_n) \approx 1.2575$$
$$\bar{\ell} \approx 1.3556 \quad (\,61/45\,)$$
$$\varrho \approx 0.0980 \quad (\,7.8\,\%\,)$$

- There are dependencies between successive source symbols !

# Lossless Coding of Sources with Memory

## Example: Stationary Markov Process

- Consider stationary Markov source $\boldsymbol{S} = \{S_n\}$ with symbol alphabet $\mathcal{A} = \{ a, b, c \}$

→ Statistically properties are completely specified by conditional pmf $p(x \,|\, y) = P(S_n = x \,|\, S_{n-1} = y)$

| $x$ | $p(x \,\|\, a)$ | $p(x \,\|\, b)$ | $p(x \,\|\, c)$ |
|---|---|---|---|
| $a$ | 0.90 | 0.15 | 0.25 |
| $b$ | 0.05 | 0.80 | 0.15 |
| $c$ | 0.05 | 0.05 | 0.60 |

calulate marginal pmf → (see exercise)

| $x$ | $p(x)$ | Huffman |
|---|---|---|
| $a$ | $29/45$ | 1 |
| $b$ | $11/45$ | 01 |
| $c$ | $5/45$ | 00 |

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- There are dependencies between successive source symbols!
- Can we exploit these dependencies for improving the coding efficiency?

# Lossless Coding of Sources with Memory

## Example: Stationary Markov Process

- Consider stationary Markov source $\boldsymbol{S} = \{S_n\}$ with symbol alphabet $\mathcal{A} = \{ a, b, c \}$

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- There are dependencies between successive source symbols !
- Can we exploit these dependencies for improving the coding efficiency ?

➡ **Idea**:
- Design Huffman code for each condition $(S_{n-1} = y)$
- Select codeword table for a symbol $s_n$ based on previous symbol $s_{n-1}$

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1}=a$" for first symbol)

| | $S_{n-1}=a$ | $S_{n-1}=b$ | $S_{n-1}=c$ |
|---|---|---|---|
| $x$ | $p(x\,\vert\,a)$ | $p(x\,\vert\,b)$ | $p(x\,\vert\,b)$ |
| $a$ | 0.90 | 0.15 | 0.25 |
| $b$ | 0.05 | 0.80 | 0.15 |
| $c$ | 0.05 | 0.05 | 0.60 |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $^{29}/_{45}$ | 0 |
| $b$ | $^{11}/_{45}$ | 10 |
| $c$ | $^{5}/_{45}$ | 11 |
| $\bar{\ell}_{\text{scal}} = {}^{61}/_{45} \approx 1.3556$ | | |

$H(S_n) \approx 1.2575$

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | $S_{n-1} = c$ |
|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | $p(x \mid b)$ |
| $a$ | 0.90 | 0 | 0.15 | 0.25 |
| $b$ | 0.05 | 10 | 0.80 | 0.15 |
| $c$ | 0.05 | 11 | 0.05 | 0.60 |

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# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | $S_{n-1} = c$ |
|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | $p(x \mid b)$ |
| $a$ | 0.90 | 0 | 0.15 | 0.25 |
| $b$ | 0.05 | 10 | 0.80 | 0.15 |
| $c$ | 0.05 | 11 | 0.05 | 0.60 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $^{29}/_{45}$ | 0 |
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# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ |
|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | | | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $^{29}/_{45}$ | 0 |
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# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ |
|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $29/45$ | 0 |
| $b$ | $11/45$ | 10 |
| $c$ | $5/45$ | 11 |
| $\bar{\ell}_{\text{scal}} = 61/45 \approx 1.3556$ | | |

$H(S_n) \approx 1.2575$

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ | |
|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | | |

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| $a$ | $^{29}/_{45}$ | 0 |
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# Conditional Huffman Codes

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| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ | |
|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $^{29}/_{45}$ | 0 |
| $b$ | $^{11}/_{45}$ | 10 |
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# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

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|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $^{29}/_{45}$ | 0 |
| $b$ | $^{11}/_{45}$ | 10 |
| $c$ | $^{5}/_{45}$ | 11 |
| $\bar{\ell}_{\text{scal}} = {}^{61}/_{45} \approx 1.3556$ | | |

$H(S_n) \approx 1.2575$

**Example**: encoding: "abba" ➡ "

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ | |
|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

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| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
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| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

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|---|---|---|---|---|---|---|
| | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

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# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

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|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
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$H(S_n) \approx 1.2575$

**Example**: encoding: "abba" ➡ "0 10

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1}=a$" for first symbol)

| | $S_{n-1}=a$ | | $S_{n-1}=b$ | | $S_{n-1}=c$ | |
|---|---|---|---|---|---|---|
| $x$ | $p(x\,\|\,a)$ | codeword | $p(x\,\|\,b)$ | codeword | $p(x\,\|\,b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $^{29}/_{45}$ | 0 |
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**Example**: encoding: "abba" ➡ "0 10 0

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ | |
|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

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**Example**: encoding: "abba" ➡ "0 10 0

$H(S_n) \approx 1.2575$

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ | |
|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

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| $a$ | $^{29}/_{45}$ | 0 |
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**Example**: encoding: "abba" ➡ "0 10 0 10"

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| $x$ | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ | |
|---|---|---|---|---|---|---|
| | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $^{29}/_{45}$ | 0 |
| $b$ | $^{11}/_{45}$ | 10 |
| $c$ | $^{5}/_{45}$ | 11 |
| $\bar{\ell}_{\text{scal}} = {}^{61}/_{45} \approx 1.3556$ | | |

$H(S_n) \approx 1.2575$

**Example**: encoding: "abba" ➡ "0 10 0 10"

decoding: "010010" ➡ ""

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ | |
|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $^{29}/_{45}$ | 0 |
| $b$ | $^{11}/_{45}$ | 10 |
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| $\bar{\ell}_{\text{scal}} = {}^{61}/_{45} \approx 1.3556$ | | |

$H(S_n) \approx 1.2575$

**Example**: encoding: "abba" ➡ "0 10 0 10"
decoding: "010010" ➡ "

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ | |
|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $^{29}/_{45}$ | 0 |
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**Example**: encoding: "abba" ➡ "0 10 0 10"
decoding: "010010" ➡ "a

# Conditional Huffman Codes

**conditional Huffman code** (assume "$S_{n-1} = a$" for first symbol)

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ | |
|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid b)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
|---|---|---|
| $a$ | $^{29}/_{45}$ | 0 |
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➡ Average codeword length $\bar{\ell}_{\text{cond}}$ for conditional Huffman code

$$\bar{\ell}_{\text{cond}} =$$

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➜ Average codeword length $\bar{\ell}_{\mathrm{cond}}$ for conditional Huffman code

$$\bar{\ell}_{\mathrm{cond}} = \sum_{y \in \mathcal{A}} p(y) \cdot \bar{\ell}_{(S_{n-1}=y)} = \frac{29}{45} \cdot 1.1 + \frac{11}{45} \cdot 1.2 + \frac{5}{45} \cdot 1.4$$

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➡ **Better than scalar Huffman code**: $\bar{\ell}_{\text{cond}} < \bar{\ell}_{\text{scal}}$

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➡ We also have: $\bar{\ell}_{\text{cond}} < H(S_n)$

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➜ **Better than scalar Huffman code**: $\bar{\ell}_{\text{cond}} < \bar{\ell}_{\text{scal}}$

➜ We also have: $\bar{\ell}_{\text{cond}} < H(S_n)$ ➜ **What's wrong ?**

# Bounds for Conditional Huffman Codes

**Consider an individual condition "$S_{n-1} = y$"** (i.e., fixed value $y$)

- Code is constructed for the conditional probability masses $p(x \,|\, y) = \mathrm{P}(S_n = x \,|\, S_{n-1} = y)$

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➡ Average codeword length $\bar{\ell}_{(S_{n-1}=y)}$ for given condition "$S_{n-1} = y$" is bounded by

$$\left( -\sum_{\forall x} p(x \mid y) \cdot \log_2 p(x \mid y) \right) \leq \bar{\ell}_{(S_{n-1}=y)} < \left( -\sum_{\forall x} p(x \mid y) \cdot \log_2 p(x \mid y) \right) + 1$$

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$$H(S_n \,|\, S_{n-1} = y) \le \bar{\ell}_{(S_{n-1}=y)} < H(S_n \,|\, S_{n-1} = y) + 1$$

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**Average codeword length $\bar{\ell}_{\mathrm{cond}}$ for conditional Huffman coding**

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## Average codeword length $\bar{\ell}_{\mathrm{cond}}$ for conditional Huffman coding

- Each condition "$S_{n-1} = y$" occurs with probability $p(y) = \mathrm{P}(S_{n-1} = y) = \mathrm{P}(S_n = y)$

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## Average codeword length $\bar{\ell}_{\mathrm{cond}}$ for conditional Huffman coding

- Each condition "$S_{n-1} = y$" occurs with probability $p(y) = \mathrm{P}(S_{n-1} = y) = \mathrm{P}(S_n = y)$

$$\left( \sum_{\forall y} p(y) \cdot H(S_n\,|\,S_{n-1} = y) \right) \leq \left( \sum_{\forall y} p(y) \cdot \bar{\ell}_{(S_{n-1}=y)} \right) < \left( \sum_{\forall y} p(y) \cdot H(S_n\,|\,S_{n-1} = y) \right) + \left( \sum_{\forall y} p(y) \cdot 1 \right)$$

# Bounds for Conditional Huffman Codes

## Consider an individual condition "$S_{n-1} = y$" (i.e., fixed value $y$)

- Code is constructed for the conditional probability masses $p(x\,|\,y) = \mathrm{P}(S_n = x\,|\,S_{n-1} = y)$

➡ Average codeword length $\bar{\ell}_{(S_{n-1}=y)}$ for given condition "$S_{n-1} = y$" is bounded by

$$\left( -\sum_{\forall x} p(x\,|\,y) \cdot \log_2 p(x\,|\,y) \right) \leq \bar{\ell}_{(S_{n-1}=y)} < \left( -\sum_{\forall x} p(x\,|\,y) \cdot \log_2 p(x\,|\,y) \right) + 1$$

$$H(S_n\,|\,S_{n-1} = y) \leq \bar{\ell}_{(S_{n-1}=y)} < H(S_n\,|\,S_{n-1} = y) + 1$$

➡ $H(S_n\,|\,S_{n-1} = y)$ is referred to as conditional entropy given the event $S_{n-1} = y$

## Average codeword length $\bar{\ell}_{\mathrm{cond}}$ for conditional Huffman coding

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$$H(S_n\,|\,S_{n-1}) \leq \bar{\ell}_{\mathrm{cond}} < H(S_n\,|\,S_{n-1}) + 1$$

# Conditional Entropy

## Conditional entropy of $S_n$ given $S_{n-1}$

$$H(S_n \,|\, S_{n-1}) = \sum_{\forall y} p(y) \cdot H(S_n \,|\, S_{n-1} = y) \qquad \text{with} \quad H(S_n|S_{n-1} = y) = -\sum_{\forall x} p(x|y) \log_2 p(x|y)$$

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$$H(S_n \mid S_{n-1}) = \sum_{\forall y} p(y) \cdot H(S_n \mid S_{n-1} = y)$$

$$= \sum_{\forall y} p(y) \left( -\sum_{\forall x} p(x \mid y) \log_2 p(x \mid y) \right)$$

with $H(S_n|S_{n-1} = y) = -\sum_{\forall x} p(x|y) \log_2 p(x|y)$

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remember: $p(x, y) = p(x \,|\, y) \, p(y)$

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&= -\sum_{\forall x, y} p(x, y) \, \log_2 p(x \,|\, y)
\end{aligned}
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similarly: $$\boxed{H(S_n) = \mathrm{E}\Big\{ -\log_2 p(S_n) \Big\}}$$

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### Conditional Huffman Codes

$$H(S_n \,|\, S_{n-1}) \le \bar{\ell} < H(S_n \,|\, S_{n-1}) + 1$$

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**Conditional Huffman Codes**

$$H(S_n \,|\, S_{n-1}) \leq \bar{\ell} < H(S_n \,|\, S_{n-1}) + 1$$

**Scalar Huffman Codes**

$$H(S_n) \leq \bar{\ell} < H(S_n) + 1$$

# Generalized Conditional Coding

## Usage of two or more preceding symbols

- Design a Huffman code for each condition: "$S_{n-1} = a,\ S_{n-2} = b,\ \cdots$"

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➡ Conditional entropy of $S_n$ given $S_{n-1}, S_{n-2}, \cdots$

$$H(S_n \,|\, S_{n-1},\, S_{n-2},\, \cdots) = \mathrm{E}\Big\{ -\log_2 p(S_n \,|\, S_{n-1}, S_{n-2}, \cdots) \Big\} = -\sum_{x, y_1, y_2, \cdots} p(x, y_1, y_2, \cdots) \cdot \log_2 p(x \,|\, y_1, y_2, \cdots)$$

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$$H(S_n \,|\, C) = \mathrm{E}\Big\{ -\log_2 p(S_n \,|\, C) \Big\} = -\sum_{x,c} p_{SC}(x, c) \cdot \log_2 p_{S|C}(x \,|\, c)$$

# Conditioning Does Not Increase Entropy

Consider general case with condition $C = f(S_{n-1}, S_{n-2}, \cdots)$

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\begin{aligned}
H(S_n \mid C) &= -\sum_{\forall s,c} p_{SC}(s,c)\, \log_2 p_{S|C}(s \mid c) \qquad \text{with} \quad p_{S|C}(s \mid c) = \frac{p_{SC}(s,c)}{p_C(c)} \\
&= -\sum_{\forall s,c} p_{SC}(s,c)\, \log_2 \left( \frac{p_{SC}(s,c)}{p_C(c)} \cdot \frac{p_S(s)}{p_S(s)} \right) \\
&= -\sum_{\forall s,c} p_{SC}(s,c)\, \log_2 p_S(s) - \sum_{\forall s,c} p_{SC}(s,c)\, \log_2 \left( \frac{p_{SC}(s,c)}{p_C(c)\, p_S(s)} \right) \\
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&= H(S_n) -
\end{aligned}
$$

# Conditioning Does Not Increase Entropy

Consider general case with condition $C = f(S_{n-1}, S_{n-2}, \cdots)$

$$H(S_n \mid C) = -\sum_{\forall s,c} p_{SC}(s,c) \log_2 p_{S|C}(s \mid c) \qquad \text{with} \quad p_{S|C}(s \mid c) = \frac{p_{SC}(s,c)}{p_C(c)}$$

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$$= H(S_n) - D(\, p_{SC} \,||\, p_S\, p_C \,)$$

# Conditioning Does Not Increase Entropy

Consider general case with condition $C = f(S_{n-1}, S_{n-2}, \cdots)$

$$
\begin{aligned}
H(S_n \,|\, C) &= -\sum_{\forall s,c} p_{SC}(s,c)\,\log_2 p_{S|C}(s\,|\,c) \qquad \text{with} \quad p_{S|C}(s\,|\,c) = \frac{p_{SC}(s,c)}{p_C(c)} \\
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&= H(S_n) - D(\,p_{SC} \,||\, p_S\,p_C\,) \qquad \text{remember:} \quad D(\,p \,||\, q\,) \ge 0
\end{aligned}
$$

# Conditioning Does Not Increase Entropy

Consider general case with condition $C = f(S_{n-1}, S_{n-2}, \cdots)$

$$H(S_n \,|\, C) = -\sum_{\forall s,c} p_{SC}(s,c)\, \log_2 p_{S|C}(s \,|\, c) \qquad \text{with} \quad p_{S|C}(s \,|\, c) = \frac{p_{SC}(s,c)}{p_C(c)}$$

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$$= H(S_n) - D(\, p_{SC} \,||\, p_S\, p_C \,)$$

remember: $D(\, p \,||\, q\,) \geq 0$

➡ $$\boxed{H(S_n \,|\, C) \leq H(S_n)}$$

# Conditioning Does Not Increase Entropy

Consider general case with condition $C = f(S_{n-1}, S_{n-2}, \cdots)$

$$
\begin{aligned}
H(S_n \,|\, C) &= -\sum_{\forall s,c} p_{SC}(s,c) \, \log_2 p_{S|C}(s \,|\, c) \qquad \text{with} \quad p_{S|C}(s \,|\, c) = \frac{p_{SC}(s,c)}{p_C(c)} \\
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&= -\sum_{\forall s} p_S(s) \, \log_2 p_S(s) - \sum_{\forall s,c} p_{SC}(s,c) \, \log_2 \left( \frac{p_{SC}(s,c)}{p_C(c)\, p_S(s)} \right) \\
&= H(S_n) - D(\, p_{SC} \,||\, p_S \, p_C \,)
\end{aligned}
$$

remember: $D(\, p \,||\, q \,) \geq 0$

➡ $\boxed{H(S_n \,|\, C) \leq H(S_n)}$ $\Big($equality if and only if $S_n$ and $C = f(S_{n-1}, S_{n-2}, \cdots)$ are independent$\Big)$

# Conditioning Does Not Increase Entropy

Consider general case with condition $C = f(S_{n-1}, S_{n-2}, \cdots)$

$$H(S_n \,|\, C) = -\sum_{\forall s,c} p_{SC}(s,c)\,\log_2 p_{S|C}(s\,|\,c) \qquad \text{with} \quad p_{S|C}(s\,|\,c) = \frac{p_{SC}(s,c)}{p_C(c)}$$

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$$= H(S_n) - D(\,p_{SC}\,||\,p_S\,p_C\,) \qquad \text{remember:} \quad D(\,p\,||\,q\,) \ge 0$$

➡ $\boxed{H(S_n \,|\, C) \le H(S_n)}$ $\quad$ (equality if and only if $S_n$ and $C = f(S_{n-1}, S_{n-2}, \cdots)$ are independent)

➡ **Conditioning does never increase entropy**

# Conditioning Does Not Increase Entropy

Consider general case with condition $C = f(S_{n-1}, S_{n-2}, \cdots)$

$$
\begin{aligned}
H(S_n \,|\, C) &= -\sum_{\forall s,c} p_{SC}(s,c)\, \log_2 p_{S|C}(s\,|\,c) \qquad\qquad \text{with} \quad p_{S|C}(s\,|\,c) = \frac{p_{SC}(s,c)}{p_C(c)} \\
&= -\sum_{\forall s,c} p_{SC}(s,c)\, \log_2 \left( \frac{p_{SC}(s,c)}{p_C(c)} \cdot \frac{p_S(s)}{p_S(s)} \right) \\
&= -\sum_{\forall s,c} p_{SC}(s,c)\, \log_2 p_S(s) - \sum_{\forall s,c} p_{SC}(s,c)\, \log_2 \left( \frac{p_{SC}(s,c)}{p_C(c)\, p_S(s)} \right) \\
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&= H(S_n) - D(\, p_{SC} \,||\, p_S\, p_C \,) \qquad\qquad \text{remember:} \quad D(\, p \,||\, q \,) \geq 0
\end{aligned}
$$

➡ $\boxed{H(S_n \,|\, C) \leq H(S_n)}$ $\Big($equality if and only if $S_n$ and $C = f(S_{n-1}, S_{n-2}, \cdots)$ are independent$\Big)$

➡ **Conditioning does never increase entropy**

- Also: **Conditional coding does never increase average codeword length**

# Example: Stationary Markov Process

## Previous Example: Stationary Markov Source

**conditional Huffman code**

| | $S_{n-1} = a$ | | $S_{n-1} = b$ | | $S_{n-1} = c$ | |
|---|---|---|---|---|---|---|
| $x$ | $p(x \mid a)$ | codeword | $p(x \mid b)$ | codeword | $p(x \mid c)$ | codeword |
| $a$ | 0.90 | 0 | 0.15 | 10 | 0.25 | 10 |
| $b$ | 0.05 | 10 | 0.80 | 0 | 0.15 | 11 |
| $c$ | 0.05 | 11 | 0.05 | 11 | 0.60 | 0 |
| | $\bar{\ell}_{(S_{n-1}=a)} = 1.1$ | | $\bar{\ell}_{(S_{n-1}=b)} = 1.2$ | | $\bar{\ell}_{(S_{n-1}=c)} = 1.4$ | |

**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
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| $a$ | $^{29}/_{45}$ | 0 |
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$\bar{\ell}_{\mathrm{scal}} \approx 1.3556$

$H(S_n) \approx 1.2575$

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➡ average codeword length: $\bar{\ell}_{\text{cond}} \approx 1.1578$

➡ conditional entropy: $H(S_n\,|\,S_{n-1}) \approx 0.7331$

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**scalar Huffman code**

| $x$ | $p(x)$ | codeword |
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$H(S_n) \approx 1.2575$

➡ Conditioning reduces entropy: from 1.2575 to 0.7331

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# Practical Example: Conditional Coding in H.264 | AVC (CAVLC)

**Table 9-5 – coeff_token mapping to TotalCoeff( coeff_token ) and TrailingOnes( coeff_token )**

| TrailingOnes ( coeff_token ) | TotalCoeff ( coeff_token ) | 0 <= nC < 2 | 2 <= nC < 4 | 4 <= nC < 8 | 8 <= nC | nC = = −1 | nC = = −2 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 11 | 1111 | 0000 11 | 01 | 1 |
| 0 | 1 | 0001 01 | 0010 11 | 0011 11 | 0000 00 | 0001 11 | 0001 111 |
| 1 | 1 | 01 | 10 | 1110 | 0000 01 | 1 | 01 |
| 0 | 2 | 0000 0111 | 0001 11 | 0010 11 | 0001 00 | 0001 00 | 0001 110 |
| 1 | 2 | 0001 00 | 0011 1 | 0111 1 | 0001 01 | 0001 10 | 0001 101 |
| 2 | 2 | 001 | 011 | 1101 | 0001 10 | 001 | 001 |
| 0 | 3 | 0000 0011 1 | 0000 111 | 0010 00 | 0010 00 | 0000 11 | 0000 0011 1 |
| 1 | 3 | 0000 0110 | 0010 10 | 0110 0 | 0010 01 | 0000 011 | 0001 100 |
| 2 | 3 | 0000 101 | 0010 01 | 0111 0 | 0010 10 | 0000 010 | 0001 011 |
| 3 | 3 | 0001 1 | 0101 | 1100 | 0010 11 | 0001 01 | 0000 1 |
| 0 | 4 | 0000 0001 11 | 0000 0111 | 0001 111 | 0011 00 | 0000 10 | 0000 0011 0 |
| 1 | 4 | 0000 0011 0 | 0001 10 | 0101 0 | 0011 01 | 0000 0011 | 0000 0010 1 |

*( continued )*

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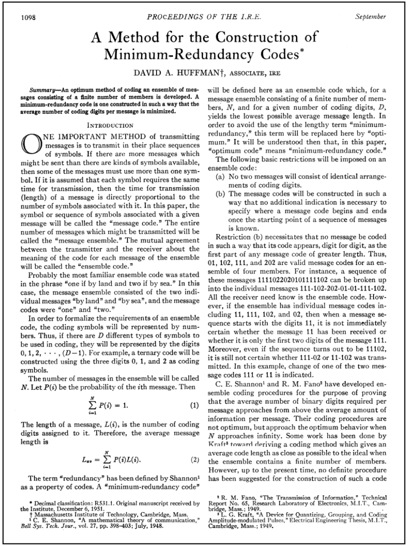

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Probably the most familiar ensemble code was stated in the phrase "one if by land and two if by sea." In this case, the message ensemble consisted of the two individual messages "by land" and "by sea", and the message codes were "one" and "two."

In order to formalize the requirements of an ensemble code, the coding symbols will be represented by numbers. Thus, if there are $D$ different types of symbols to be used in coding, they will be represented by the digits $0, 1, 2, \cdots, (D-1)$. For example, a ternary code will be constructed using the three digits 0, 1, and 2 as coding symbols.

The number of messages in the ensemble will be called $N$. Let $P(i)$ be the probability of the $i$th message. Then

$$\sum_{i=1}^{N} P(i) = 1. \qquad (1)$$

The length of a message, $L(i)$, is the number of coding digits assigned to it. Therefore, the average message length is

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The term "redundancy" has been defined by Shannon[1] as a property of codes. A "minimum-redundancy code" will be defined here as an ensemble code which, for a message ensemble consisting of a finite number of members, $N$, and for a given number of coding digits, $D$, yields the lowest possible average message length. In order to avoid the use of the lengthy term "minimum-redundancy," this term will be replaced here by "optimum." It will be understood then that, in this paper, "optimum code" means "minimum-redundancy code."

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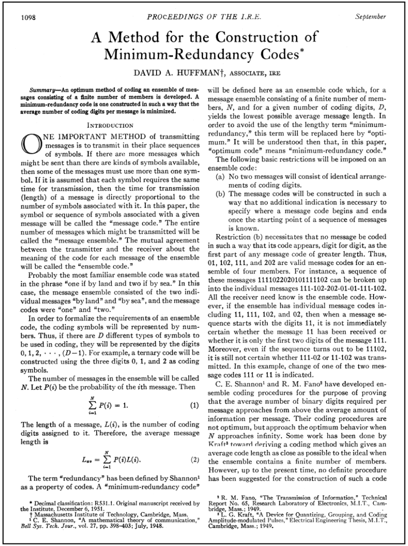

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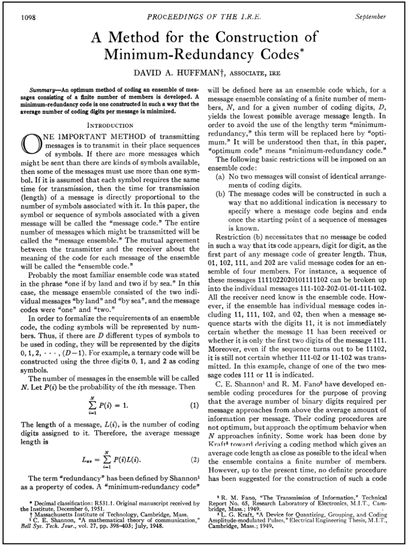

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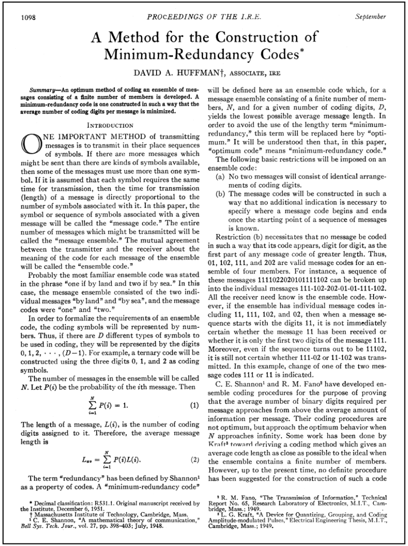

1098 PROCEEDINGS OF THE I.R.E. September

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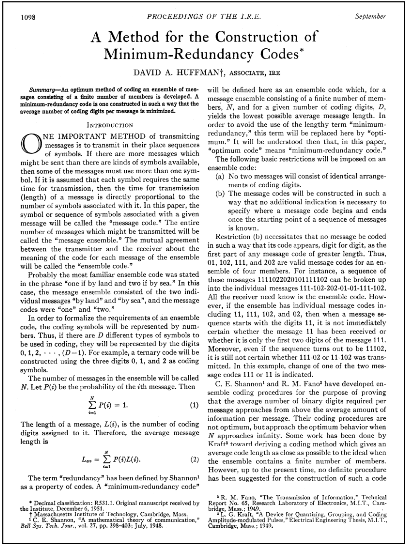

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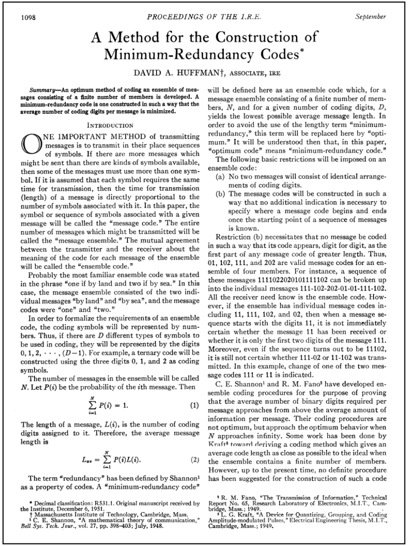


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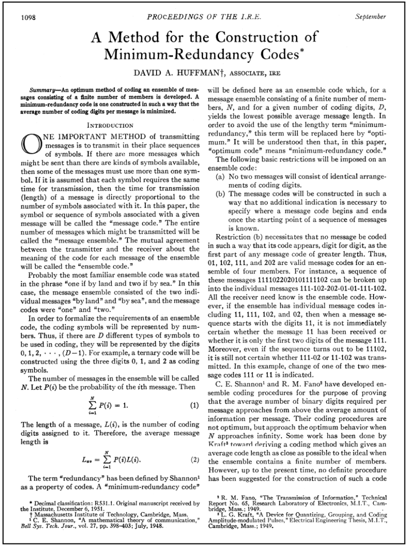

1098 *PROCEEDINGS OF THE I.R.E.* *September*

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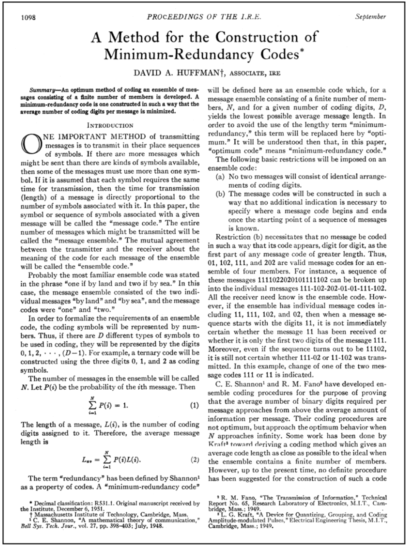

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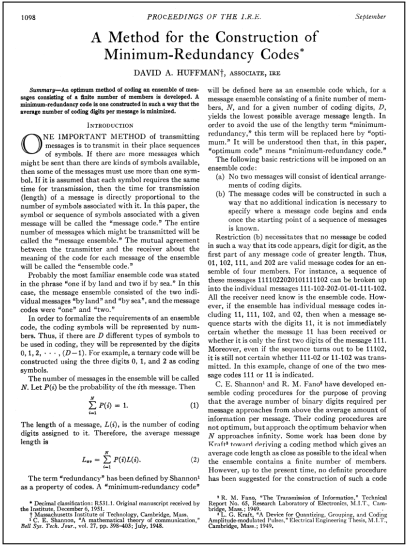

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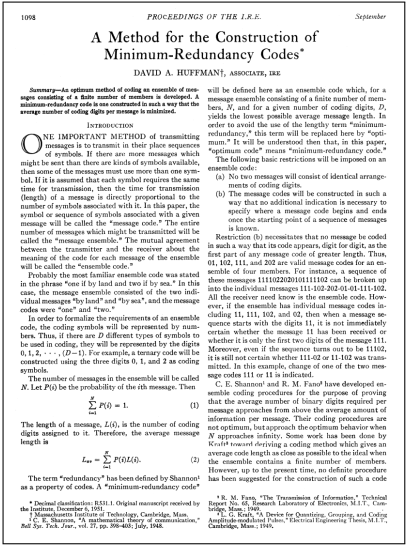

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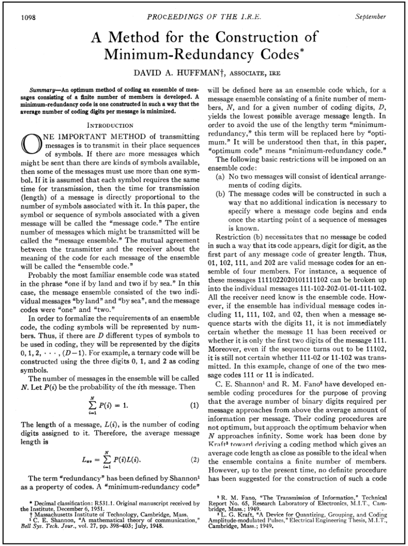

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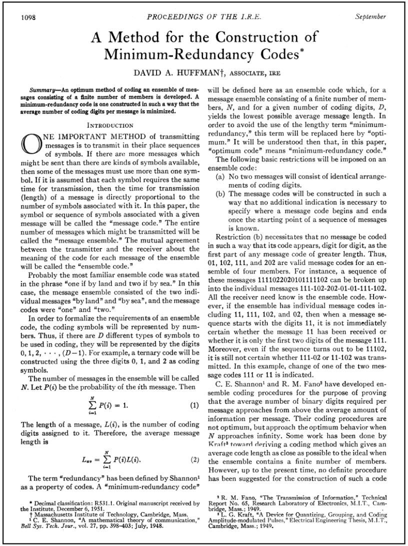

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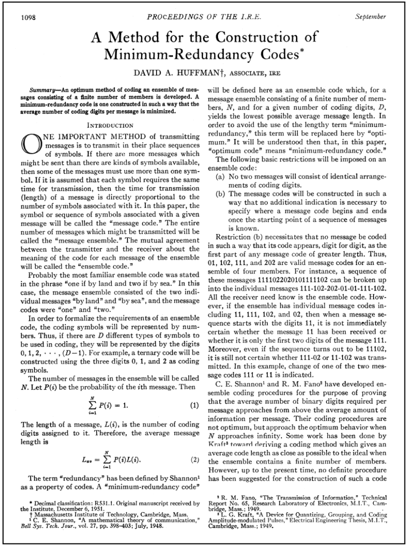

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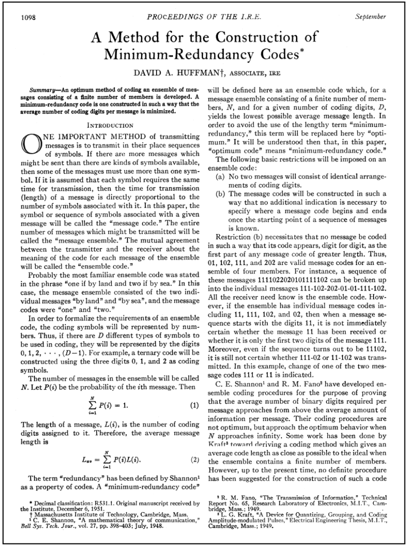

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The term "redundancy" has been defined by Shannon[1] as a property of codes. A "minimum-redundancy code" will be defined here as an ensemble code which, for a message ensemble consisting of a finite number of members, $N$, and for a given number of coding digits, $D$, yields the lowest possible average message length. In order to avoid the use of the lengthy term "minimum-redundancy," this term will be replaced here by "optimum." It will be understood then that, in this paper, "optimum code" means "minimum-redundancy code."

The following basic restrictions will be imposed on an ensemble code:

(a) No two messages will consist of identical arrangements of coding digits.

(b) The message codes will be constructed in such a way that no additional indication is necessary to specify where a message code begins and ends once the starting point of a sequence of messages is known.

Restriction (b) necessitates that no message be coded in such a way that its code appears, digit for digit, as the first part of any message code of greater length. Thus, 01, 102, 111, and 202 are valid message codes for an ensemble of four members. For instance, a sequence of these messages 1111022020101111102 can be broken up into the individual messages 111-102-202-01-01-111-102. All the receiver need know is the ensemble code. However, if the ensemble has individual message codes including 11, 111, 102, and 02, then when a message sequence starts with the digits 11, it is not immediately certain whether the message 11 has been received or whether it is only the first two digits of the message 111. Moreover, even if the sequence turns out to be 11102, it is still not certain whether 111-02 or 11-102 was transmitted. In this example, change of one of the two message codes 111 or 11 is indicated.

C. E. Shannon[1] and R. M. Fano[2] have developed ensemble coding procedures for the purpose of proving that the average number of binary digits required per message approaches from above the average amount of information per message. Their coding procedures are not optimum, but approach the optimum behavior when $N$ approaches infinity. Some work has been done by Kraft[3] toward deriving a coding method which gives an average code length as close as possible to the ideal when the ensemble contains a finite number of members. However, up to the present time, no definite procedure has been suggested for the construction of such a code

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# Example: Binary Markov Process

## Black and White Document Scans

- Binary random process $\boldsymbol{S} = \{S_n\}$:

$$S_n = 0 \quad \rightarrow \quad \text{white sample}$$
$$S_n = 1 \quad \rightarrow \quad \text{black sample}$$

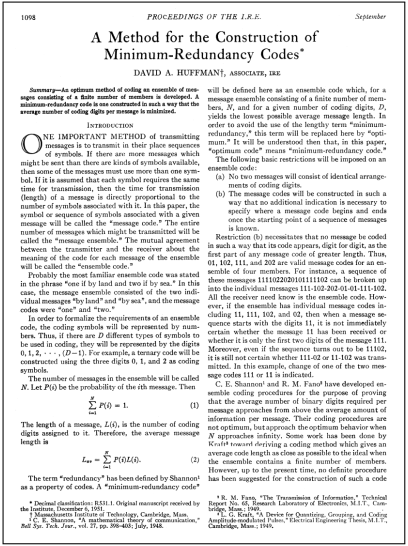
- Statistics measured over a large set of documents

$$p(0) \;=\; 0.8$$
$$p(0\,|\,0) \;=\; 0.9$$

➡ Model: Stationary Markov process

➡ Determine remaining probabilities

$$p(1) = 1 - p(0) = 0.2$$
$$p(1\,|\,0) = 1 - p(0\,|\,0) = 0.1$$
$$p(0\,|\,1) = \frac{p(1\,|\,0) \cdot p(0)}{p(1)} = 0.4$$
$$p(1\,|\,1) = 1 - p(0\,|\,1) = 0.6$$



1098 PROCEEDINGS OF THE I.R.E. September

### A Method for the Construction of Minimum-Redundancy Codes*

DAVID A. HUFFMAN†, ASSOCIATE, IRE

*Summary*—**An optimum method of coding an ensemble of messages consisting of a finite number of members is developed. A minimum-redundancy code is one constructed in such a way that the average number of coding digits per message is minimized.**

INTRODUCTION

ONE IMPORTANT METHOD of transmitting messages is to transmit in their place sequences of symbols. If there are more messages which might be sent than there are kinds of symbols available, then some of the messages must use more than one symbol. If it is assumed that each symbol requires the same time for transmission, then the time for transmission (length) of a message is directly proportional to the number of symbols associated with it. In this paper, the symbol or sequence of symbols associated with a given message will be called the "message code." The entire number of messages which might be transmitted will be called the "message ensemble." The mutual agreement between the transmitter and the receiver about the meaning of the code for each message of the ensemble will be called the "ensemble code."

Probably the most familiar ensemble code was stated in the phrase "one if by land and two if by sea." In this case, the message ensemble consisted of the two individual messages "by land" and "by sea", and the message codes were "one" and "two."

In order to formalize the requirements of an ensemble code, the coding symbols will be represented by numbers. Thus, if there are $D$ different types of symbols to be used in coding, they will be represented by the digits $0, 1, 2, \cdots, (D-1)$. For example, a ternary code will be constructed using the three digits 0, 1, and 2 as coding symbols.

The number of messages in the ensemble will be called $N$. Let $P(i)$ be the probability of the $i$th message. Then

$$\sum_{i=1}^{N} P(i) = 1. \tag{1}$$

The length of a message, $L(i)$, is the number of coding digits assigned to it. Therefore, the average message length is

$$L_{av} = \sum_{i=1}^{N} P(i)L(i). \tag{2}$$

The term "redundancy" has been defined by Shannon[1] as a property of codes. A "minimum-redundancy code" will be defined here as an ensemble code which, for a message ensemble consisting of a finite number of members, $N$, and for a given number of coding digits, $D$, yields the lowest possible average message length. In order to avoid the use of the lengthy term "minimum-redundancy," this term will be replaced here by "optimum." It will be understood then that, in this paper, "optimum code" means "minimum-redundancy code."

The following basic restrictions will be imposed on an ensemble code:

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# Example: Scalar and Conditional Coding

**conditional Huffman code**

| | $S_{n-1} = 0$ | $S_{n-1} = 1$ |
|---|---|---|
| $x$ | $p(x \mid 0)$ | $p(x \mid 1)$ |
| 0 | 0.9 | 0.4 |
| 1 | 0.1 | 0.6 |

**scalar Huffman code**

| $x$ | $p(x)$ |
|---|---|
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| 1 | 0.2 |

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$H(S_n) \approx 0.7219$

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- Conditioning does not improve coding efficiency, even though $H(S_n\,|\,S_{n-1}) < H(S_n)$

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**How can we improve coding efficiency ?**

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**How can we improve coding efficiency ? ➡ Joint coding of multiple symbols ?**

# Block Huffman Codes

## Block Codes

- Code $N > 1$ successive symbols jointly (using a single codeword)

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➡ Design code for $N$-dimensional joint pmf $p(x_1, x_2, \cdots, x_N) = \mathrm{P}(S_1 = x_1, S_2 = x_2, \cdots, S_N = x_N)$

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### Block Huffman Coding for Black and White Document Scans

$N = 2$ symbols

| $s_1 s_2$ | $p(s_1, s_2)$ | codewords |
|---|---|---|
| 00 | | |
| 01 | | |
| 10 | | |
| 11 | | |

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| 11 | 0.12 | |

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$\bar{\ell}_2 = 1.44$

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$N = 3$ symbols

| $s_1 s_2 s_3$ | $p(s_1, s_2, s_3)$ | codewords |
|---|---|---|
| 000 | | |
| 001 | | |
| 010 | | |
| 011 | | |
| 100 | | |
| 101 | | |
| 110 | | |
| 111 | | |

# Block Huffman Codes

## Block Codes

- Code $N > 1$ successive symbols jointly (using a single codeword)
- ➡ Design code for $N$-dimensional joint pmf $p(x_1, x_2, \cdots, x_N) = \mathrm{P}(S_1 = x_1, S_2 = x_2, \cdots, S_N = x_N)$
- ➡ Optimal block code: Huffman algorithm

### Block Huffman Coding for Black and White Document Scans

$N = 2$ symbols

| $s_1 s_2$ | $p(s_1, s_2)$ | codewords |
|---|---|---|
| 00 | 0.72 | 1 |
| 01 | 0.08 | 010 |
| 10 | 0.08 | 011 |
| 11 | 0.12 | 00 |

$$\bar{\ell}_2 = 1.44$$
$$\bar{\ell} = \bar{\ell}_2 / 2 = 0.72$$

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| $s_1 s_2 s_3$ | $p(s_1, s_2, s_3)$ | codewords |
|---|---|---|
| 000 | 0.648 | |
| 001 | 0.072 | |
| 010 | 0.032 | |
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$\bar{\ell}_3 = 1.952$

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- Similar to marginal entropy, but for blocks of $N$ symbols

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### Bounds for Block Huffman Codes

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→ Average codeword length $\bar{\ell}$ per symbol: $\dfrac{H_N(\boldsymbol{S})}{N} \le \bar{\ell} < \dfrac{H_N(\boldsymbol{S})}{N} + \dfrac{1}{N}$

# Chain Rule for Entropies

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# Increasing Block Size Does Not Increases Lower Bound

- Chain rule of entropies

$$H_N(\boldsymbol{S}) \;=\; H(S_1) + H(S_2 \,|\, S_1) + H(S_3 \,|\, S_1, S_2) + \ldots + H(S_N \,|\, S_1, S_2, \cdots, S_{N-1})$$

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&\geq N \cdot H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \qquad \text{(conditioning does not increase entropy)}
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- Consider one step of chain rule

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H_N(S_1, S_2, \cdots, S_N) = H_{N-1}(S_1, S_2, \cdots, S_{N-1}) + H(S_N \,|\, S_1, S_2, \cdots, S_{N-1})
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$$N \cdot H_N(\boldsymbol{S}) = N \cdot H_{N-1}(\boldsymbol{S}) + N \cdot H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \qquad \text{(multiplied by } N\text{)}$$

# Increasing Block Size Does Not Increases Lower Bound

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# Increasing Block Size Does Not Increases Lower Bound

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# Increasing Block Size Does Not Increases Lower Bound

- Chain rule of entropies

$$\begin{aligned} H_N(\boldsymbol{S}) &= H(S_1) + H(S_2 \,|\, S_1) + H(S_3 \,|\, S_1, S_2) + \ldots + H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \\ &\geq N \cdot H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \qquad \text{(conditioning does not increase entropy)} \end{aligned}$$

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$$\begin{aligned} H_N(S_1, S_2, \cdots, S_N) &= H_{N-1}(S_1, S_2, \cdots, S_{N-1}) + H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \\ N \cdot H_N(\boldsymbol{S}) &= N \cdot H_{N-1}(\boldsymbol{S}) + N \cdot H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \qquad \text{(multiplied by } N\text{)} \\ N \cdot H_N(\boldsymbol{S}) &\leq N \cdot H_{N-1}(\boldsymbol{S}) + H_N(\boldsymbol{S}) \qquad \text{(applied above inequality)} \\ (N-1) \cdot H_N(\boldsymbol{S}) &\leq N \cdot H_{N-1}(\boldsymbol{S}) \end{aligned}$$

➡ **Increasing block size never increases lower bound**

$$\boxed{\frac{H_N(\boldsymbol{S})}{N} \leq \frac{H_{N-1}(\boldsymbol{S})}{N-1}} \qquad \Big(\text{equality if and only if } \boldsymbol{S} \text{ is iid: } H(S_N \,|\, S_{N-1}, \cdots) = H(S_N)\Big)$$

# Block Huffman Coding with Increasing Block Sizes

## Example: Stationary Markov Source

**conditional pmf**

| $x$ | $p(x\|a)$ | $p(x\|b)$ | $p(x\|c)$ |
|---|---|---|---|
| $a$ | 0.90 | 0.15 | 0.25 |
| $b$ | 0.05 | 0.80 | 0.15 |
| $c$ | 0.05 | 0.05 | 0.60 |

**scalar Huffman coding**:

$$\bar{\ell}_{\mathrm{scal}} = 1.3556$$

**conditional Huffman coding**:

$$\bar{\ell}_{\mathrm{cond}} = 1.1578$$

# Block Huffman Coding with Increasing Block Sizes

## Example: Stationary Markov Source

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| $x$ | $p(x\|a)$ | $p(x\|b)$ | $p(x\|c)$ |
|---|---|---|---|
| $a$ | 0.90 | 0.15 | 0.25 |
| $b$ | 0.05 | 0.80 | 0.15 |
| $c$ | 0.05 | 0.05 | 0.60 |

**block Huffman coding**

| $N$ | $\frac{H_N(\boldsymbol{S})}{N}$ | $\bar{\ell} = \frac{\bar{\ell}_N}{N}$ | number of codewords |
|---|---|---|---|
| 1 | 1.2575 | 1.3556 | 3 |

**scalar Huffman coding**:

$$\bar{\ell}_{\mathrm{scal}} = 1.3556$$

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**block Huffman coding**

| $N$ | $\frac{H_N(\boldsymbol{S})}{N}$ | $\bar{\ell} = \frac{\bar{\ell}_N}{N}$ | number of codewords |
|---|---|---|---|
| 1 | 1.2575 | 1.3556 | 3 |
| 2 | 0.9953 | 1.0094 | 9 |

# Block Huffman Coding with Increasing Block Sizes

## Example: Stationary Markov Source

**conditional pmf**

| $x$ | $p(x\|a)$ | $p(x\|b)$ | $p(x\|c)$ |
|---|---|---|---|
| $a$ | 0.90 | 0.15 | 0.25 |
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| $N$ | $\frac{H_N(\boldsymbol{S})}{N}$ | $\bar{\ell} = \frac{\bar{\ell}_N}{N}$ | number of codewords |
|---|---|---|---|
| 1 | 1.2575 | 1.3556 | 3 |
| 2 | 0.9953 | 1.0094 | 9 |
| 3 | 0.9079 | 0.9150 | 27 |

# Block Huffman Coding with Increasing Block Sizes

## Example: Stationary Markov Source

**conditional pmf**

| $x$ | $p(x\|a)$ | $p(x\|b)$ | $p(x\|c)$ |
|---|---|---|---|
| $a$ | 0.90 | 0.15 | 0.25 |
| $b$ | 0.05 | 0.80 | 0.15 |
| $c$ | 0.05 | 0.05 | 0.60 |

**scalar Huffman coding**:

$$\bar{\ell}_{\text{scal}} = 1.3556$$

**conditional Huffman coding**:

$$\bar{\ell}_{\text{cond}} = 1.1578$$

**block Huffman coding**

| $N$ | $\frac{H_N(\boldsymbol{S})}{N}$ | $\bar{\ell} = \frac{\bar{\ell}_N}{N}$ | number of codewords |
|---|---|---|---|
| 1 | 1.2575 | 1.3556 | 3 |
| 2 | 0.9953 | 1.0094 | 9 |
| 3 | 0.9079 | 0.9150 | 27 |
| 4 | 0.8642 | 0.8690 | 81 |
| 5 | 0.8380 | 0.8462 | 243 |
| 6 | 0.8205 | 0.8299 | 729 |
| 7 | 0.8080 | 0.8153 | 2187 |
| 8 | 0.7987 | 0.8027 | 6561 |
| 9 | 0.7914 | 0.7940 | 19683 |

# Example: Two Successive Quantization Indexes in MP3

## Block Huffman Coding in MP3

- Joint coding of two successive quantization indexes
- Multiple Huffman tables specified in standard (designed offline)

**Huffman code table 5**

| x | y | hlen | hcod |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 3 | 010 |
| 0 | 2 | 6 | 000110 |
| 0 | 3 | 7 | 0000101 |
| 1 | 0 | 3 | 011 |
| 1 | 1 | 3 | 001 |
| 1 | 2 | 6 | 000100 |
| 1 | 3 | 7 | 0000100 |
| 2 | 0 | 6 | 000111 |
| 2 | 1 | 6 | 000101 |
| 2 | 2 | 7 | 0000111 |
| 2 | 3 | 8 | 00000001 |
| 3 | 0 | 7 | 0000110 |
| 3 | 1 | 6 | 000001 |
| 3 | 2 | 7 | 0000001 |
| 3 | 3 | 8 | 00000000 |

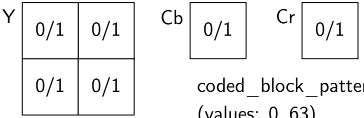
**Huffman code table 6**

| x | y | hlen | hcod |
|---|---|---|---|
| 0 | 0 | 3 | 111 |
| 0 | 1 | 3 | 011 |
| 0 | 2 | 5 | 00101 |
| 0 | 3 | 7 | 0000001 |
| 1 | 0 | 3 | 110 |
| 1 | 1 | 2 | 10 |
| 1 | 2 | 4 | 0011 |
| 1 | 3 | 5 | 00010 |
| 2 | 0 | 4 | 0101 |
| 2 | 1 | 4 | 0100 |
| 2 | 2 | 5 | 00100 |
| 2 | 3 | 6 | 000001 |
| 3 | 0 | 6 | 000011 |
| 3 | 1 | 5 | 00011 |
| 3 | 2 | 6 | 000010 |
| 3 | 3 | 7 | 0000000 |

**Huffman code table 8**

| x | y | hlen | hcod |
|---|---|---|---|
| 0 | 0 | 2 | 11 |
| 0 | 1 | 3 | 100 |
| 0 | 2 | 6 | 000110 |
| 0 | 3 | 8 | 00010010 |
| 0 | 4 | 8 | 00001100 |
| 0 | 5 | 9 | 000000101 |
| 1 | 0 | 3 | 101 |
| 1 | 1 | 2 | 01 |
| 1 | 2 | 4 | 0010 |
| 1 | 3 | 8 | 00010000 |
| 1 | 4 | 8 | 00001001 |
| 1 | 5 | 8 | 00000011 |
| 2 | 0 | 6 | 000111 |
| 2 | 1 | 4 | 0011 |
| 2 | 2 | 6 | 000101 |
| 2 | 3 | 8 | 00001110 |
| 2 | 4 | 8 | 00000111 |
| 2 | 5 | 9 | 000000011 |
| 3 | 0 | 8 | 00010011 |
| 3 | 1 | 8 | 00010001 |
| 3 | 2 | 8 | 00001111 |
| 3 | 3 | 9 | 000001101 |
| 3 | 4 | 9 | 000001010 |
| 3 | 5 | 10 | 0000000100 |
| 4 | 0 | 8 | 00001101 |
| 4 | 1 | 7 | 0000101 |
| 4 | 2 | 8 | 00001000 |
| 4 | 3 | 9 | 000001011 |
| 4 | 4 | 10 | 0000000101 |
| 4 | 5 | 10 | 0000000001 |
| 5 | 0 | 9 | 000001100 |
| 5 | 1 | 8 | 00000100 |
| 5 | 2 | 9 | 000000100 |
| 5 | 3 | 9 | 000000001 |
| 5 | 4 | 11 | 00000000001 |
| 5 | 5 | 11 | 00000000000 |

# Example: Coded Block Pattern in MPEG-2




coded_block_pattern = xxxxxx (bit mask)
(values: 0..63)

**Table B.9 – Variable length codes for coded_block_pattern**

| coded_block_pattern VLC code | cbp | coded_block_pattern VLC code | cbp |
|---|---|---|---|
| 111 | 60 | 0001 1100 | 35 |
| 1101 | 4 | 0001 1011 | 13 |
| 1100 | 8 | 0001 1010 | 49 |
| 1011 | 16 | 0001 1001 | 21 |
| 1010 | 32 | 0001 1000 | 41 |
| 1001 1 | 12 | 0001 0111 | 14 |
| 1001 0 | 48 | 0001 0110 | 50 |
| 1000 1 | 20 | 0001 0101 | 22 |
| 1000 0 | 40 | 0001 0100 | 42 |
| 0111 1 | 28 | 0001 0011 | 15 |
| 0111 0 | 44 | 0001 0010 | 51 |

(*continued*)

# Lossless Source Coding Theorem

## Entropy Rate

- Observation: Lower bound for block coding typically decreases with increasing $N$

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➡ **Entropy rate**: Limit ($N \to \infty$) for lower bound of block coding

$$\bar{H}(\boldsymbol{S}) = \lim_{N\to\infty} \frac{H(S_0, \cdots, S_{N-1})}{N} = \lim_{N\to\infty} \frac{H_N(\boldsymbol{S})}{N}$$

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- The limit always exists for stationary random sources

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- The limit always exists for stationary random sources

## Fundamental Lossless Source Coding Theorem

➜ Average codeword length for all lossless codes is bounded by

$$\bar{\ell} \geq \bar{H}(\boldsymbol{S}) = \lim_{N\to\infty} \frac{H_N(\boldsymbol{S})}{N}$$

# Lossless Source Coding Theorem

## Entropy Rate

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➡ **Entropy rate**: Limit ($N \to \infty$) for lower bound of block coding

$$\bar{H}(\boldsymbol{S}) = \lim_{N\to\infty} \frac{H(S_0, \cdots, S_{N-1})}{N} = \lim_{N\to\infty} \frac{H_N(\boldsymbol{S})}{N}$$

- The limit always exists for stationary random sources

## Fundamental Lossless Source Coding Theorem

➡ Average codeword length for all lossless codes is bounded by

$$\bar{\ell} \geq \bar{H}(\boldsymbol{S}) = \lim_{N\to\infty} \frac{H_N(\boldsymbol{S})}{N}$$

- Asymptotically achievable with block Huffman coding for $N \to \infty$

# Entropy Rate for IID Sources

$$\bar{H}(\boldsymbol{S}) = \lim_{N\to\infty} \frac{1}{N} H(S_1, S_2, \cdots, S_N)$$

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$$= \lim_{N\to\infty} \frac{1}{N} \Big( H(S_1) + H(S_2 \,|\, S_1) + H(S_3 \,|\, S_1, S_2) + \cdots + H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \Big)$$

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$$= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} H(S_k)$$

# Entropy Rate for IID Sources

$$
\begin{aligned}
\bar{H}(\boldsymbol{S}) &= \lim_{N\to\infty} \frac{1}{N} H(S_1, S_2, \cdots, S_N) \\
&= \lim_{N\to\infty} \frac{1}{N} \Big( H(S_1) + H(S_2 \,|\, S_1) + H(S_3 \,|\, S_1, S_2) + \cdots + H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \Big) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} H(S_k) \\
&= \lim_{N\to\infty} \frac{1}{N} \Big( N \cdot H(S) \Big)
\end{aligned}
$$

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&= H(S)
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$$

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$$\begin{aligned}
\bar{H}(\boldsymbol{S}) &= \lim_{N\to\infty} \frac{1}{N}\, H(S_1, S_2, \cdots, S_N) \\
&= \lim_{N\to\infty} \frac{1}{N} \Big( H(S_1) + H(S_2 \,|\, S_1) + H(S_3 \,|\, S_1, S_2) + \cdots + H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \Big) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} H(S_k) \\
&= \lim_{N\to\infty} \frac{1}{N} \Big( N \cdot H(S) \Big) \\
&= H(S)
\end{aligned}$$

Note: Block Huffman coding may still improve coding efficiency

# Entropy Rate for Stationary Markov Sources

$$\bar{H}(\boldsymbol{S}) = \lim_{N\to\infty} \frac{1}{N} H(S_1, S_2, \cdots, S_N)$$

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\end{aligned}
$$

# Entropy Rate for Stationary Markov Sources

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&= \lim_{N\to\infty} \frac{1}{N} \left( H(S_1) + \sum_{k=2}^{N} H(S_k \,|\, S_{k-1}) \right)
\end{aligned}
$$

# Entropy Rate for Stationary Markov Sources

$$
\begin{aligned}
\bar{H}(\boldsymbol{S}) &= \lim_{N\to\infty} \frac{1}{N}\, H(S_1, S_2, \cdots, S_N) \\
&= \lim_{N\to\infty} \frac{1}{N} \Big( H(S_1) + H(S_2 \,|\, S_1) + H(S_3 \,|\, S_1, S_2) + \cdots + H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \Big) \\
&= \lim_{N\to\infty} \frac{1}{N} \left( H(S_1) + \sum_{k=2}^{N} H(S_k \,|\, S_{k-1}) \right) \\
&= \lim_{N\to\infty} \frac{1}{N} \Big( H(S) + (N-1) \cdot H(S_n \,|\, S_{n-1}) \Big)
\end{aligned}
$$

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$$
\begin{aligned}
\bar{H}(\boldsymbol{S}) &= \lim_{N\to\infty} \frac{1}{N}\, H(S_1, S_2, \cdots, S_N) \\
&= \lim_{N\to\infty} \frac{1}{N} \Big( H(S_1) + H(S_2 \,|\, S_1) + H(S_3 \,|\, S_1, S_2) + \cdots + H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \Big) \\
&= \lim_{N\to\infty} \frac{1}{N} \left( H(S_1) + \sum_{k=2}^{N} H(S_k \,|\, S_{k-1}) \right) \\
&= \lim_{N\to\infty} \frac{1}{N} \Big( H(S) + (N-1) \cdot H(S_n \,|\, S_{n-1}) \Big) \\
&= \lim_{N\to\infty} \frac{H(S)}{N} + \lim_{N\to\infty} \frac{N-1}{N} \cdot H(S_n \,|\, S_{n-1})
\end{aligned}
$$

# Entropy Rate for Stationary Markov Sources

$$
\begin{aligned}
\bar{H}(\boldsymbol{S}) &= \lim_{N\to\infty} \frac{1}{N}\, H(S_1, S_2, \cdots, S_N) \\
&= \lim_{N\to\infty} \frac{1}{N} \Big( H(S_1) + H(S_2 \,|\, S_1) + H(S_3 \,|\, S_1, S_2) + \cdots + H(S_N \,|\, S_1, S_2, \cdots, S_{N-1}) \Big) \\
&= \lim_{N\to\infty} \frac{1}{N} \left( H(S_1) + \sum_{k=2}^{N} H(S_k \,|\, S_{k-1}) \right) \\
&= \lim_{N\to\infty} \frac{1}{N} \Big( H(S) + (N-1) \cdot H(S_n \,|\, S_{n-1}) \Big) \\
&= \lim_{N\to\infty} \frac{H(S)}{N} + \lim_{N\to\infty} \frac{N-1}{N} \cdot H(S_n \,|\, S_{n-1}) \\
&= H(S_n \,|\, S_{n-1})
\end{aligned}
$$

# Summary of Lecture: Huffman Codes

## Huffman Algorithm

- Generates prefix codes with minimum redundancy for any given pmf
- Yields optimal uniquely decodable codes

## Scalar Huffman codes

- Codeword table assigns codeword to each individual symbol
- Table size is equal to alphabet size

## Conditional Huffman codes

- Separate codeword table for each possible condition (e.g., preceding symbol)
- Switch codeword tables during encoding and decoding

## Block Huffman codes

- Code $N > 1$ successive symbols jointly
- Codeword table assigns codeword to combination of $N$ successive symbols

# Summary of Lecture: Entropy Measures and Bounds

## Entropy Measures

- Scalar (or marginal) entropy: $H(S) = -\sum_a p(a) \cdot \log_2 p(a)$
- Conditional entropy: $H(S\,|\,C) = -\sum_{a,c} p(a,c) \cdot \log_2 p(a\,|\,c)$
- Block entropy: $H_N(\boldsymbol{S}) = -\sum_{x_1,\cdots,x_N} p(x_1,\cdots,x_N) \cdot \log_2 p(x_1,\cdots,x_N)$
- Entropy rate: $\bar{H}(\boldsymbol{S}) = \lim_{N\to\infty} \frac{1}{N} H_N(\boldsymbol{S})$

## Bounds for Lossless Coding

- Scalar Huffman coding: $H(S) \le \bar{\ell} < H(S) + 1$
- Conditional Huffman coding: $H(S\,|\,C) \le \bar{\ell} < H(S\,|\,C) + 1$
- Block Huffman coding: $\frac{1}{N} H_N(\boldsymbol{S}) \le \bar{\ell} < \frac{1}{N} H_N(\boldsymbol{S}) + \frac{1}{N}$
- **All lossless codes**: $\boxed{\bar{\ell} \ge \bar{H}(\boldsymbol{S})}$ (**fundamental lossless source coding theorem**)

# Exercise 1: Huffman Code

Given is a discrete iid process $\mathbf{X}$ with the alphabet $\mathcal{A} = \{a, b, c, d, e, f, g\}$. The pmf $p_X(x)$ is specified in the following table.

| $x$ | $p_X(x)$ |
|---|---|
| $a$ | 1/3 |
| $b$ | 1/9 |
| $c$ | 1/27 |
| $d$ | 1/27 |
| $e$ | 1/27 |
| $f$ | 1/9 |
| $g$ | 1/3 |

**(a)** Develop a Huffman code for the given pmf $p_X(x)$.

**(b)** Calculate the average codeword length of the developed Huffman code.

**(c)** Calculate the absolute and relative redundancy for the developed Huffman code.

# Exercise 2: Huffman Codes and Entropy Measures

Let $\boldsymbol{Z} = \{Z_n\}$ be a binary iid process with alphabet $\{0, 1\}$ and pmf $\{0.5, 0.5\}$ (e.g., coin toss).

Let $\boldsymbol{X} = \{X_n\}$ be a random process given by $X_n = Z_{n-1} + Z_n$.

**(a)** Determine the marginal pmf $p_X(x)$ and the marginal entropy $H(X)$.

**(b)** Develop a scalar Huffman code and calculate its average codeword length.

**(c)** Determine the conditional pmf $p_{X_n|X_{n-1}}(x_n \,|\, x_{n-1})$ and the conditional entropy $H(X_n \,|\, X_{n-1})$.

**(d)** Develop a conditional Huffman code and calculate its average codeword length.

**(e)** Develop a block Huffman code for $N = 2$ symbols and calculate its average codeword length.

**(f)** Optional (more difficult):

- Derive a formula for the $N$-th order block entropy $H_N(X_n, \cdots, X_{n+N-1})$.
- Determine the entropy rate $\bar{H}(\boldsymbol{X})$.
- Is $\boldsymbol{X}$ a Markov process?

# Exercise 3: Estimate Entropy Measures (Implementation Task)

Write a program (in a programming language of your choice) that estimates the following entropy measures based on the statistics of a given input file:

- Marginal entropy: $H(S_n)$
- 1-st order conditional entropy: $H(S_n \,|\, S_{n-1})$
- Block entropy of size $N = 2$: $H(S_n, S_{n+1})$ [ calculate also $H(S_n, S_{n+1})/2$ ]

Assume that all files represent a sequence of 8-bit samples (i.e., each byte represents a symbol).

Test your program for the following sample files (from course website):

- white uniform noise: "whiteUniformNoise.raw"
- white Gaussian noise: "whiteGaussianNoise.raw"
- correlated Gaussian noise: "correlatedGaussianNoise.raw"
- English text file: "englishText.txt"
- 8-bit audio data: "audioData.raw"
- 8-bit image data: "imageData.raw"

What can you conclude about the potential to compress these files?

# Exercise 4: Geometric Pmf (Optional)

Given is a Bernoulli process (binary iid process) $\mathbf{B} = \{B_n\} = \{B\}$ with the alphabet $\mathcal{A}_B = \{0, 1\}$ and the pmf $p_B(0) = p$, $p_B(1) = 1 - p$ with $0 < p < 1$.

Consider the random variable $X$ that specifies the number of successive random variables $B_n$ that have to be observed to get exactly one "1".

**(a)** Determine the pmf for $X$ as function of $p$.

**(b)** Determine the entropy $H(B)$ as function of p.

**(c)** Determine the entropy $H(X)$ as function of $H(B)$ and $p$.

**(d)** What structure has an optimal scalar variable-length code for $X$ and $p \leq 0.5$?
Calculate its average codeword length as function of p.
Calculate its relative redundancy as function of p.

*Hints*:

$$\forall_{|a|<1}, \sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \qquad \text{and} \qquad \forall_{|a|<1}, \sum_{k=0}^{\infty} k\, a^k = \frac{a}{(1-a)^2}$$