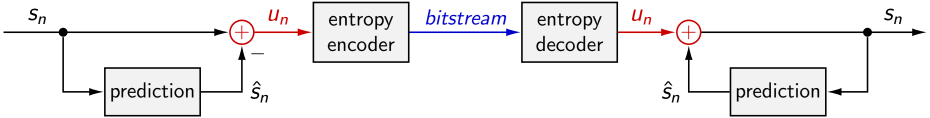
Predictive Lossless Coding



Last Lecture: Arithmetic Coding

Arithmetic Coding

- Practical realization of Shannon-Fano-Elias Coding (using standard integer arithmetic)
- No codeword table, on-the-fly encoding and decoding

Arithmetic Coding vs Huffman Coding

- For given block size N: Huffman coding is optimal, arithmetic coding is suboptimal
- Arithmetic coding is realizable for large N, while Huffman coding is not

Coding Efficiency of Arithmetic Coding

- Given probabilities and large N: Coding efficiency is very close to theoretical optimum
- In practice: Coding efficiency depends on using suitable probabilities
  - ➡ Adaptive pmf estimation during encoding and decoding
  - ➡ Using conditional pmfs (switch adaptive pmfs during encoding and decoding)

Arithmetic Coding with Adaptive Pmfs

Basic Idea

- Update pmf after encoding of each symbol
- Update pmf after decoding of each symbol

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Straightforward realization

- Count occurrences N_k of alphabet letters a_k
- V-bit probabilities $p_V(a_k)$ are given by

$$p_V(a_k) = \left\lfloor 2^V \cdot \frac{N_k}{\sum_k N_k} \right\rfloor \qquad \left(\begin{array}{l} \text{rounding} \\ \text{down !} \end{array} \right)$$

- Initialization: $\forall k, \quad N_k = 1$

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```
class Pmf // example for adaptive pmf implementation
{
public:
  Pmf( int Vbits, int numLetters, int maxSumCounts )
  : V          ( Vbits )
  , maxSum     ( maxSumCounts   ) // adaption speed
  , sumCounts( numLetters      )
  , counts     ( numLetters, 1 ) // all counts = 1
  {}

  int operator[] ( int index ) const  {
    return ( counts[ index ] << V ) / sumCounts;
  }

  void update( int index )  {
    counts[ index ]++;
    if( ++sumCounts >= maxSum ) {
      sumCounts = 0;
      for( auto& cnt : counts )   // rounding up !
        sumCounts += ( cnt = ( cnt + 1 ) >> 1 );
    }
  }

private:
  const int    V;             // number of bits for pmf
  const int    maxSum;        // maximum sum of counts
  int          sumCounts;     // sum of all counts
  vector<int>  counts;        // counts for letters
};
```

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- Two (or more) directly preceding samples
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```
void encodeMessage( const vector<int>& message, ... )
{
  vector<Pmf>  pmfs( numLetters, {...} );
  ArithEncoder aenc( ... );
  int lastSymbol = 0;
  for( const auto& currSymbol : message )  {
    Pmf& currPmf   = pmfs[ lastSymbol ];
    aenc.encode    ( currSymbol, currPmf );
    currPmf.update( currSymbol );
    lastSymbol     = currSymbol;
  }
  aenc.terminate();
}
```

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```
vector<int> decodeMessage( int numSymbols, ... )
{
  vector<int>  message;
  vector<Pmf>  pmfs( numLetters, {...} );
  ArithDecoder adec( ... );
  int symbol = 0;
  while( message.size() < numSymbols )  {
    Pmf& currPmf   = pmfs[ symbol ];
    symbol         = adec.decode( currPmf );
    currPmf.update    ( symbol );
    message.push_back( symbol );
  }
  return message;
}
```

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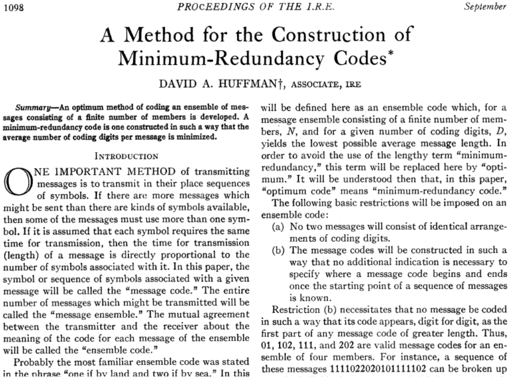

1098 PROCEEDINGS OF THE I.R.E. September

A Method for the Construction of Minimum-Redundancy Codes*

DAVID A. HUFFMAN†, ASSOCIATE, IRE

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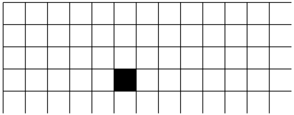

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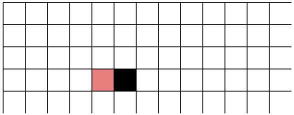

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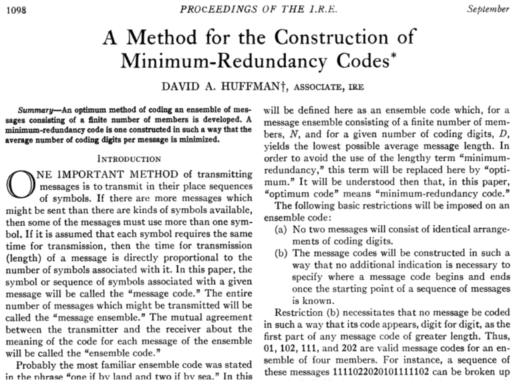

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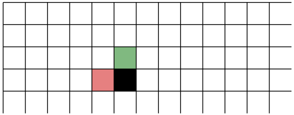

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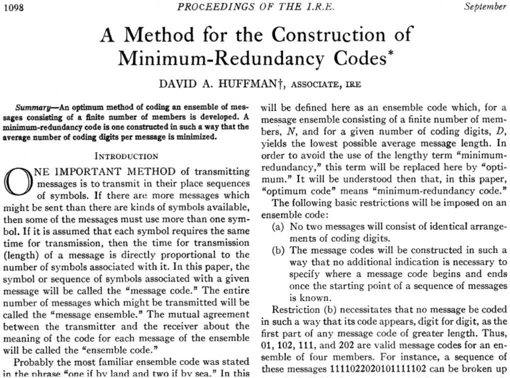

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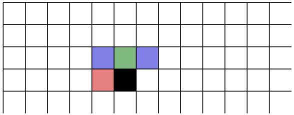

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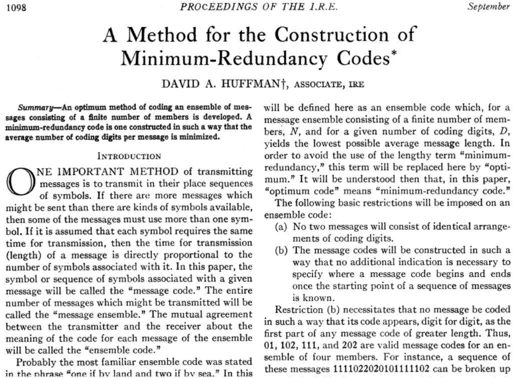

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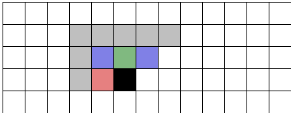

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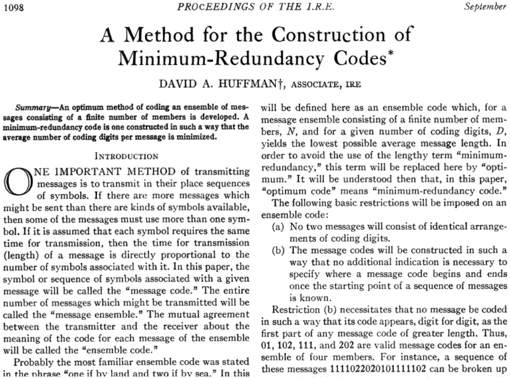

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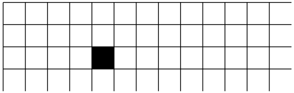

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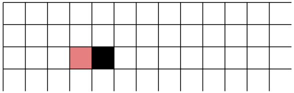

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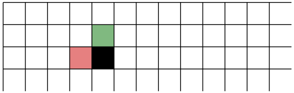

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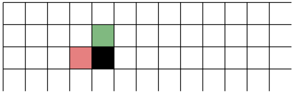
3 Left sample and above sample:
- ➜ 16 777 216 probability masses ($2^8 \cdot 2^8 \cdot 2^8$)
- ➜ File size = 221 849 bytes (84.62 %)

Original:	262 159	bytes	(512 × 512)
gzip:	222 999	bytes	(85.06 %)
bzip2:	173 877	bytes	(66.33 %)
lzip:	180 000	bytes	(68.66 %)

Example: Conditional Arithmetic Coding for 8-bit Images



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gzip:	222 999	bytes	(85.06 %)
bzip2:	173 877	bytes	(66.33 %)
lzip:	180 000	bytes	(68.66 %)

1 No condition:
- ➡ 256 probability masses (2^8)
- ➡ File size = 240 112 bytes (91.59 %)

2 Left sample:
- ➡ 65 536 probability masses ($2^8 \cdot 2^8$)
- ➡ File size = 179 179 bytes (68.35 %)

3 Left sample and above sample:
- ➡ 16 777 216 probability masses ($2^8 \cdot 2^8 \cdot 2^8$)
- ➡ File size = 221 849 bytes (84.62 %)

- ➡ **Too many probability masses**
- ➡ **Pmfs do not adapt to image statistics**

Example: Conditional Arithmetic Coding for 16-bit Audio

Original:	26 559 960	bytes	(5:01 minutes)
gzip:	24 926 843	bytes	(93.85 %)
bzip2:	22 445 509	bytes	(84.51 %)
lzip:	23 777 258	bytes	(89.52 %)

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- ➡ 4 294 967 296 probability masses ($2^{16} \cdot 2^{16}$)

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- ➡ **Requires 16 GByte of memory (when we use 32 bit per probability)**
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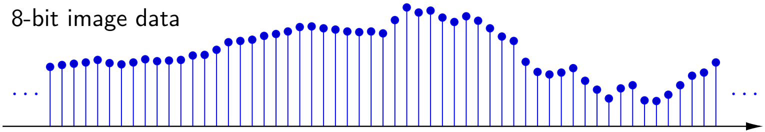
➡ **Cannot exploit dependencies between symbols using this type of coding !!!**

Analysis: Typical Properties of Real Signals



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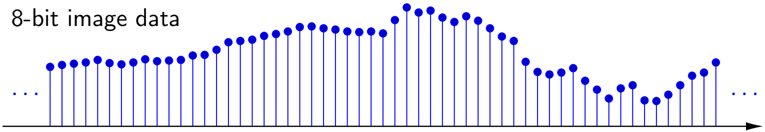
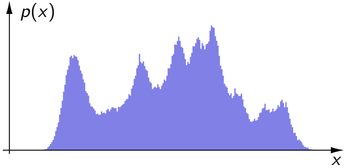


Marginal Pmf

- Not much room for compression
- Entropy $H = 7.45$ (8-bit data)



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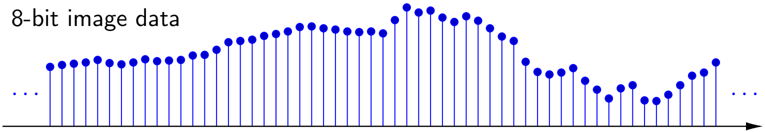

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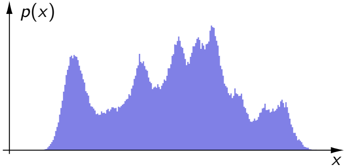
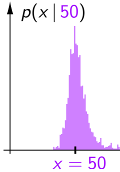
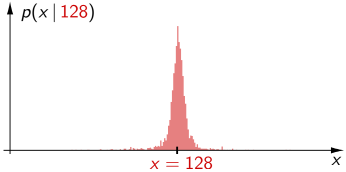


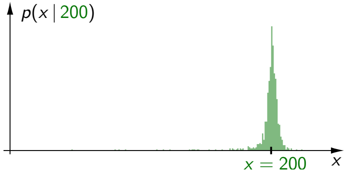
Conditional Pmfs

- Significantly narrower than marginal pmf (room for compression)
- Shape is very similar for all conditions, but shifted by x_{n-1}



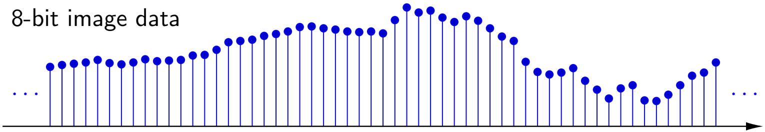
$$p(x \mid a) = \mathrm{P}(X_n = x \mid X_{n-1} = a)$$

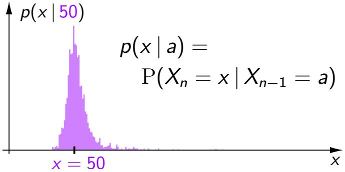




Analysis: Typical Properties of Real Signals

8-bit image data





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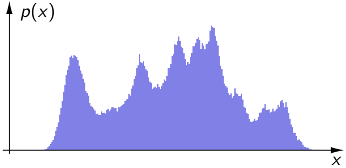
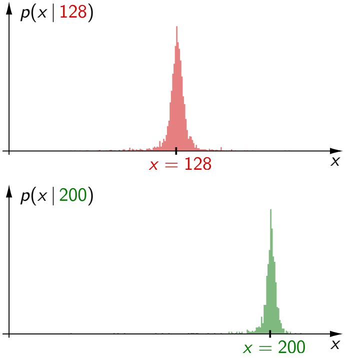


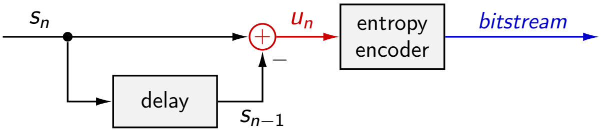


Conditional Pmfs

- Significantly narrower than marginal pmf (room for compression)
- Shape is very similar for all conditions, but shifted by x_{n-1}

➡ **Idea: Code difference to previous sample !!!**

Simple Predictive Coding using Preceding Sample



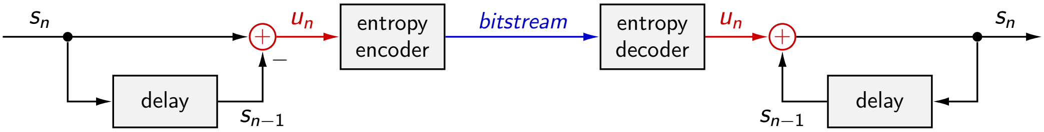
Encoding

- Generate difference sample

$$u_n = s_n - s_{n-1}$$

- Encode difference sample u_n

Simple Predictive Coding using Preceding Sample



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Decoding

- Decode difference sample u_n
- Reconstruct original sample

$$s_n = s_{n-1} + u_n$$

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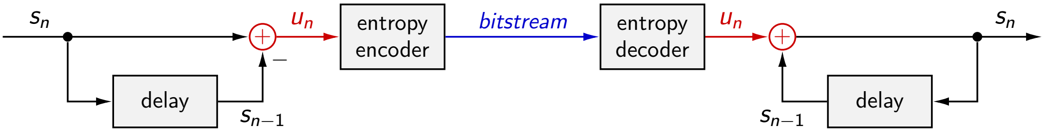

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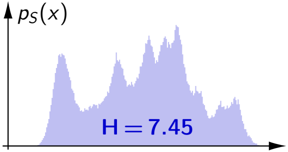
Basic Effect

- Prediction removes large part of inter-symbol dependencies before actual entropy coding
- Reduces required complexity for the entropy coding (e.g., marginal instead of conditional coding)

How Well Does That Type of Prediction Work ?



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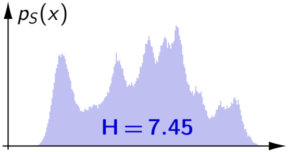

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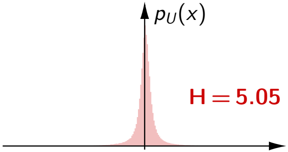

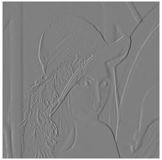

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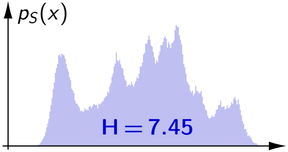
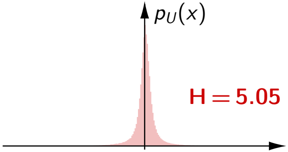
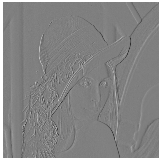
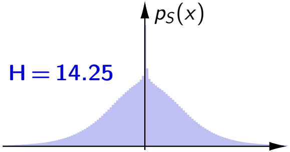


$$u_n = s_n - s_{n-1} \longrightarrow$$







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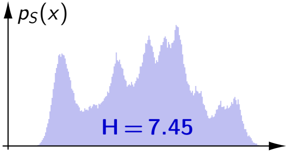

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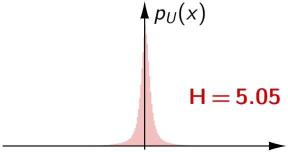

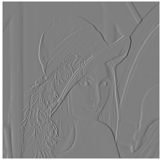

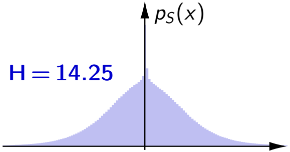

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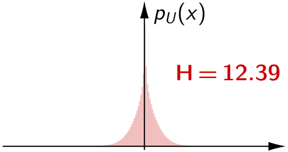
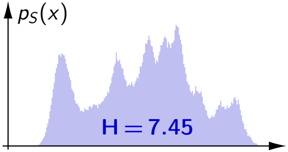


saves nearly 2 bits per sample !

How Well Does That Type of Prediction Work ?



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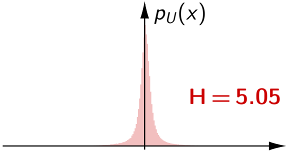

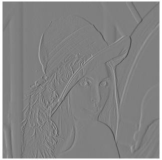

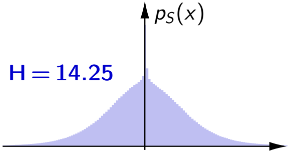

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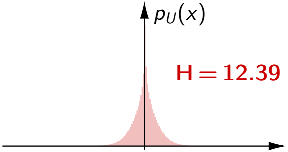

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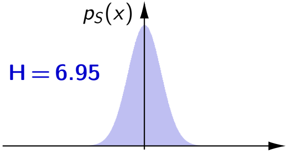
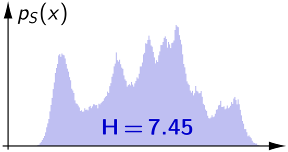
White Gaussian Noise



How Well Does That Type of Prediction Work ?



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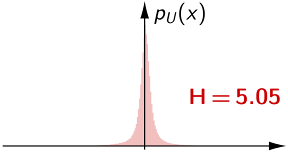

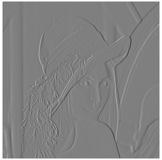

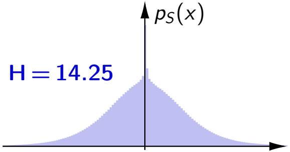

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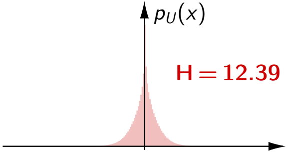

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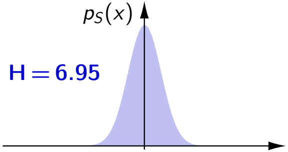

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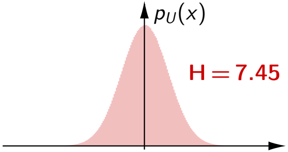
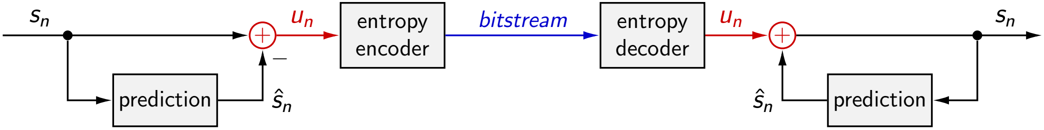


prediction increases entropy !

General Predictive Coding



Predictive Lossless Coding

- Predict current sample s_n using a function of preceding samples

$$\hat{s}_n = f(s_{n-1}, s_{n-2}, \cdots)$$

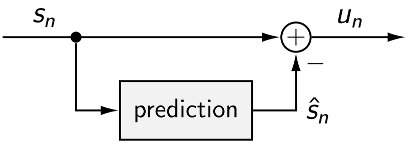
- Entropy coding of prediction error samples

$$u_n = s_n - \hat{s}_n$$

- Decoder uses exactly the same prediction and reconstructs the original samples according to

$$s_n = \hat{s}_n + u_n$$

Choice of Observation Set and Predictor

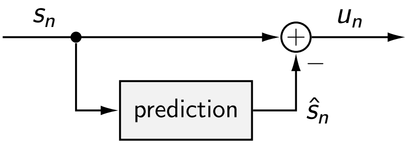


Choice of Observation Set $\mathcal{B}_n$

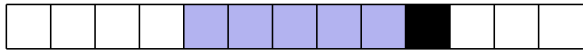
- Use small number of preceding samples for prediction

➡ Choose the samples $\mathcal{B}_n$ with highest dependencies to current sample s_n

Choice of Observation Set and Predictor



1D signals:



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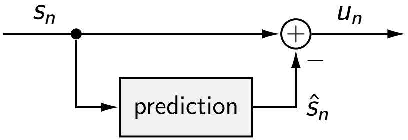
  - 1D signals (e.g., audio): N directly preceding samples $\rightarrow$ $\mathcal{B}_n = \{s_{n-1}, s_{n-2}, \cdots, s_{n-N}\}$

Choice of Observation Set and Predictor



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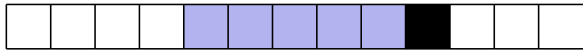
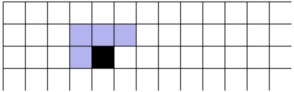


2D signals:



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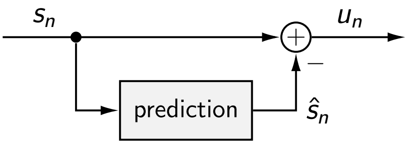
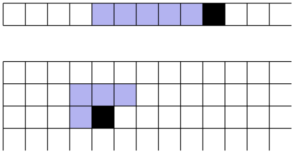
- 1D signals (e.g., audio): N directly preceding samples $\rightarrow$ $\mathcal{B}_n = \{s_{n-1}, s_{n-2}, \cdots, s_{n-N}\}$
- 2D signals (e.g., images): Samples in causal direct neighborhood

Choice of Observation Set and Predictor





Choice of Observation Set $\mathcal{B}_n$

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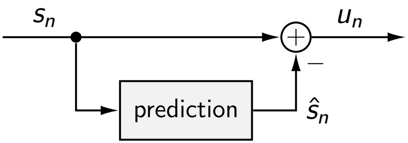
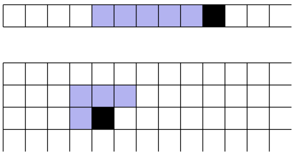
- 2D signals (e.g., images): Samples in causal direct neighborhood

Choice of Predictor for given Observation Set

- Question: What function $\hat{s}_n = f(\mathcal{B}_n)$ should we use for prediction ?

Choice of Observation Set and Predictor





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- Use small number of preceding samples for prediction

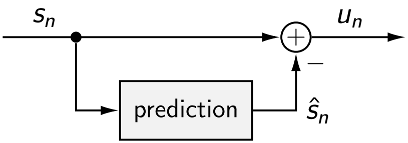
➡ Choose the samples $\mathcal{B}_n$ with highest dependencies to current sample s_n

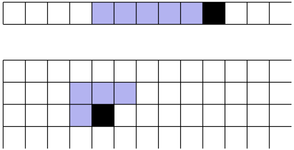
- 1D signals (e.g., audio): N directly preceding samples $\rightarrow$ $\mathcal{B}_n = \{s_{n-1}, s_{n-2}, \cdots, s_{n-N}\}$
- 2D signals (e.g., images): Samples in causal direct neighborhood

Choice of Predictor for given Observation Set

- Question: What function $\hat{s}_n = f(\mathcal{B}_n)$ should we use for prediction ?
- Want to minimize entropy $H(U_n)$ of prediction error samples $\{u_n\}$ (difficult to access directly)

Choice of Observation Set and Predictor





Choice of Observation Set $\mathcal{B}_n$

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➡ Can minimize variance σ_U^2 and ignore shape of pmf $p(u)$

Optimization Criterion for Deriving Predictor

Typical Optimization Criterion

- Minimize Energy of Prediction Error Signal

$$\varepsilon_U^2 = \mathrm{E}\{\, U_n^2 \,\} = \mathrm{E}\left\{ \left(S_n - \hat{S}_n \right)^2 \right\} = \mathrm{E}\left\{ \left(S_n - f(\mathcal{B}_n) \right)^2 \right\}$$

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- Reformulate prediction error energy

$$\begin{aligned} \varepsilon_U^2 &= \mathrm{E}\{\, U_n^2 \,\} = \mathrm{E}\Big\{ (U_n - \mu_U + \mu_u)^2 \Big\} \\ &= \mathrm{E}\Big\{ (U_n - \mu_U)^2 \Big\} + \mu_U^2 + 2\,\mu_U\,\mathrm{E}\{\, U_n - \mu_U \,\} \\ &= \sigma_U^2 + \mu_U^2 \end{aligned}$$

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➡ Minimization of squared prediction error ε_U^2 implies minimization of variance σ_U^2 and mean μ_U^2

➡ Minimize the width of the pmf $p_U(u)$, but ignore its actual shape

➡ Suitable alternative to minimization of the entropy $H(U)$

Optimal Predictor for Given Observation Set

- Question: What value a minimizes the prediction error energy ?

$$\varepsilon_U^2 = \mathrm{E}\{\,(S_n - a)^2\,\}$$

Optimal Predictor for Given Observation Set

- Question: What value *a* minimizes the prediction error energy ?

$$\varepsilon_U^2 \;=\; \mathrm{E}\{\,(S_n - a)^2\,\} = \mathrm{E}\{\,(S_n - \mathrm{E}\{\,S_n\,\} + \mathrm{E}\{\,S_n\,\} - a)^2\,\}$$

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&= \sigma_S^2 + \underbrace{(\mathrm{E}\{\, S_n \,\} - a)^2}_{\geq 0}
\end{aligned}
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Minimization of mean-squared prediction error for given observation set $\mathcal{B}_n$

- Similar optimization problem: Minimize $\mathrm{E}\Big\{\,(S_n - f(\mathcal{B}_n))^2\,|\,\mathcal{B}_n\,\Big\}$

Optimal Predictor for Given Observation Set

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➔ Solution: **Optimal predictor is given by the conditional mean**

$$
\boxed{\hat{s}_n = f_{\mathrm{opt}}(\mathcal{B}_n) = \mathrm{E}\{ S_n \,|\, \mathcal{B}_n \}}
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➞ General case requires storage of large tables (often impractical and complex)

Optimal Predictor for Autoregressive Sources

Autoregressive Sources of Order m

- Good probabilistic model for many real signals: **AR(m) process**

$$S_n = \mu_s + \sum_{k=1}^{m} a_k \cdot (S_{n-k} - \mu_s) + Z_n \qquad (Z_n \text{ is zero-mean iid process})$$

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➡ Optimal predictor if we know the entire past, i.e., $\mathcal{B}_n = \{s_{n-1}, s_{n-2}, \cdots\}$

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&= a_0 + \sum_{k=1}^{m} a_k \cdot s_{n-k} \qquad \text{with} \quad a_0 = \mu_s \left(1 - \sum_{k=1}^{m} a_k \right)
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➡ **Optimal predictor is an affine function of the past m samples**

Variance and Mean for Affine Prediction

Affine Predictor: $$\boxed{\hat{s}_n = a_0 + \sum_{k=1}^{K} a_k \cdot b_k}$$ for any observation set $\mathcal{B}_n = \{b_1, b_2, \cdots, b_K\}$

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- **Mean μ_U of prediction error**

$$\mu_U^2 = \mathrm{E}\{\, U_n \,\}$$

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- **Variance σ_U^2 of predictor error** does not depend on constant offset a_0

$$\sigma_U^2 \;=\; \mathrm{E}\{\, (U_n - \mathrm{E}\{\, U_n \,\})^2 \,\}$$

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$$= \mathrm{E}\Big\{ \Big(\Big(S_n - \textstyle\sum_{k=1}^{K} a_k\, B_k \Big) - \mu_S \Big(1 - \sum_{k=1}^{K} a_k\Big)\Big)^2 \Big\} \neq f(a_0)$$

Affine and Linear Prediction

Affine Predictor: $$\boxed{\hat{s}_n = a_0 + \sum_{k=1}^{K} a_k \cdot b_k}$$ for any observation set $\mathcal{B}_n = \{b_1, b_2, \cdots, b_K\}$

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➡ **How can we derive optimal prediction parameters $\{a_1, a_2, \cdots, a_K\}$ for a given observation set and a given source ?**

Consider: Linear Prediction and Minimization of Variance

Vector Notation (for simplifying derivations)

- Observation set: $\boldsymbol{B}_n = (B_1, B_2, \cdots, B_K)^{\mathrm{T}}$

Consider: Linear Prediction and Minimization of Variance

Vector Notation (for simplifying derivations)

- Observation set: $\boldsymbol{B}_n = (B_1, B_2, \cdots, B_K)^{\mathrm{T}}$
- Prediction parameters: $\boldsymbol{a} = (a_1, a_2, \cdots, a_K)^{\mathrm{T}}$

Consider: Linear Prediction and Minimization of Variance

Vector Notation (for simplifying derivations)

- Observation set: $\boldsymbol{B}_n = (B_1, B_2, \cdots, B_K)^{\mathrm{T}}$
- Prediction parameters: $\boldsymbol{a} = (a_1, a_2, \cdots, a_K)^{\mathrm{T}}$

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Optimization Problem

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$$\sigma_U^2(\boldsymbol{a}) = \mathrm{E}\Big\{ \big(U_n - \mathrm{E}\{ U_n \} \big)^2 \Big\}$$

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$$\sigma_U^2(\boldsymbol{a}) = \mathrm{E}\left\{ \Big(U_n - \mathrm{E}\{\, U_n \,\} \Big)^2 \right\} = \mathrm{E}\left\{ \Big(S_n - \boldsymbol{a}^{\mathrm{T}} \boldsymbol{B}_n - \mathrm{E}\{\, S_n - \boldsymbol{a}^{\mathrm{T}} \boldsymbol{B}_n \,\} \Big)^2 \right\}$$

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$$
\begin{aligned}
\sigma_U^2(\boldsymbol{a}) &= \mathrm{E}\Big\{ \big(U_n - \mathrm{E}\{ U_n \} \big)^2 \Big\} = \mathrm{E}\Big\{ \big(S_n - \boldsymbol{a}^{\mathrm{T}} \boldsymbol{B}_n - \mathrm{E}\{ S_n - \boldsymbol{a}^{\mathrm{T}} \boldsymbol{B}_n \} \big)^2 \Big\} \\
&= \mathrm{E}\Big\{ \Big(\big(S_n - \mathrm{E}\{ S_n \} \big) - \boldsymbol{a}^{\mathrm{T}} \big(\boldsymbol{B}_n - \mathrm{E}\{ \boldsymbol{B}_n \} \big) \Big)^2 \Big\}
\end{aligned}
$$

Reformulate Prediction Error Variance

- Prediction error variance

$$\sigma_U^2(\boldsymbol{a}) \;=\; \mathrm{E}\left\{ \left(\left(S_n - \mathrm{E}\{\, S_n \,\} \right) - \boldsymbol{a}^{\mathrm{T}} \left(\boldsymbol{B}_n - \mathrm{E}\{\, \boldsymbol{B}_n \,\} \right) \right)^2 \right\}$$

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&= \mathrm{E}\left\{ \left(S_n - \mathrm{E}\{\, S_n \,\} \right)^2 \right\} - 2\, \boldsymbol{a}^{\mathrm{T}} \cdot \mathrm{E}\left\{ \left(S_n - \mathrm{E}\{\, S_n \,\} \right) \left(\boldsymbol{B}_n - \mathrm{E}\{\, \boldsymbol{B}_n \,\} \right) \right\} \\
&\quad + \boldsymbol{a}^{\mathrm{T}} \cdot \mathrm{E}\left\{ \left(\boldsymbol{B}_n - \mathrm{E}\{\, \boldsymbol{B}_n \,\} \right) \left(\boldsymbol{B}_n - \mathrm{E}\{\, \boldsymbol{B}_n \,\} \right)^{\mathrm{T}} \right\} \cdot \boldsymbol{a}
\end{aligned}
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&= \sigma_S^2 - 2\, \boldsymbol{a}^{\mathrm{T}}\, \boldsymbol{c} + \boldsymbol{a}^{\mathrm{T}}\, \boldsymbol{C}_B\, \boldsymbol{a}
\end{aligned}$$

with $\boldsymbol{C}_B$ and $\boldsymbol{c}$ being given by

$$\begin{aligned}
\boldsymbol{C}_B &= \mathrm{E}\Big\{ \big(\boldsymbol{B}_n - \mathrm{E}\{ \boldsymbol{B}_n \} \big) \big(\boldsymbol{B}_n - \mathrm{E}\{ \boldsymbol{B}_n \} \big)^{\mathrm{T}} \Big\} \\
\boldsymbol{c} &= \mathrm{E}\Big\{ \big(S_n - \mathrm{E}\{ S_n \} \big) \big(\boldsymbol{B}_n - \mathrm{E}\{ \boldsymbol{B}_n \} \big) \Big\}
\end{aligned}$$

Auto-Covariance Matrix of the Observation Set

$$\boldsymbol{C}_B = \mathrm{E}\Big\{ \big(\boldsymbol{B}_n - \mathrm{E}\{\,\boldsymbol{B}_n\,\}\big)\big(\boldsymbol{B}_n - \mathrm{E}\{\,\boldsymbol{B}_n\,\}\big)^{\mathrm{T}} \Big\} = \sigma_S^2 \cdot \begin{bmatrix} 1 & \varrho_{1,2} & \varrho_{1,3} & \cdots & \varrho_{1,K} \\ \varrho_{2,1} & 1 & \varrho_{2,3} & \cdots & \varrho_{2,K} \\ \varrho_{3,1} & \varrho_{3,2} & 1 & \cdots & \varrho_{3,K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho_{K,1} & \varrho_{K,2} & \varrho_{K,3} & \cdots & 1 \end{bmatrix}$$

- Each entry represents the correlation coefficient between two samples of the observation set

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➡ Property of the source (and choice of observation set)

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Cross-Covariance Vector of the Observation Set and Current Sample

$$\boldsymbol{c} = \mathrm{E}\Big\{ \big(S_n - \mathrm{E}\{ S_n \}\big)\big(\boldsymbol{B}_n - \mathrm{E}\{ \boldsymbol{B}_n \}\big) \Big\} = \sigma_S^2 \cdot \begin{bmatrix} \varrho_{c,1} \\ \varrho_{c,2} \\ \varrho_{c,3} \\ \vdots \\ \varrho_{c,K} \end{bmatrix}$$

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- Goal: Minimization of prediction error variance σ_U^2

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➡ Optimal affine prediction: Additionally, set

$$a_0 = \mu_S \left(1 - \sum_{k=1}^{K} a_k \right)$$

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4. Determine constant offset

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Example: Prediction of Audio Signals using Three Preceding Samples

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Example

$$\mu_S = -37.6917$$
$$\varrho_1 = 0.9581$$
$$\varrho_2 = 0.8619$$
$$\varrho_3 = 0.7564$$

Example: Prediction of Audio Signals using Three Preceding Samples

3 Solve linear equation system

$$\begin{bmatrix} 1 & \varrho_1 & \varrho_2 \\ \varrho_1 & 1 & \varrho_1 \\ \varrho_2 & \varrho_1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \varrho_1 \\ \varrho_2 \\ \varrho_3 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{aligned} a_1 &= 1.9409 \\ a_2 &= -1.4580 \\ a_3 &= 0.4804 \end{aligned}$$

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$$a_0 = \mu_S \left(1 - \sum_{k=1}^{K} a_k \right) \qquad \Longrightarrow \qquad a_0 = -1.3833$$

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$$\begin{aligned} \hat{s}_n &= \mathrm{round}\Big(a_0 + a_1 \cdot s_{n-1} + a_2 \cdot s_{n-2} + a_3 \cdot s_{n-3} \Big) \\ &= \mathrm{round}\Big(-1.3833 + 1.9409 \cdot s_{n-1} - 1.4580 \cdot s_{n-2} + 0.4804 \cdot s_{n-3} \Big) \end{aligned}$$

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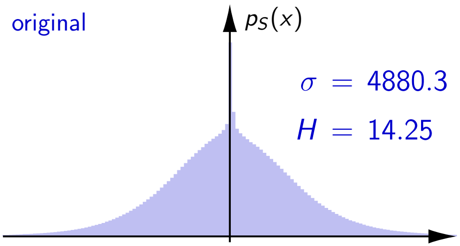
Note: Predictor must be rounded to an integer (want to apply lossless coding)

Example: Prediction of Audio Signals using Three Preceding Samples



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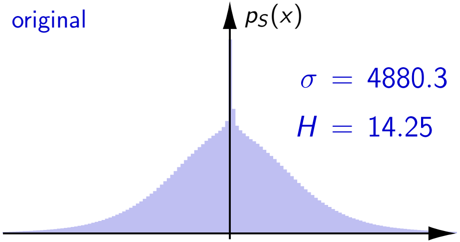

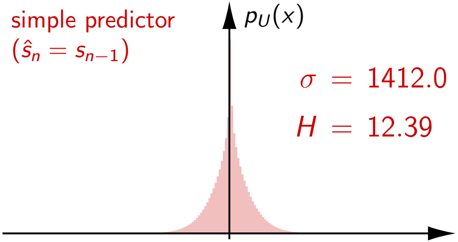

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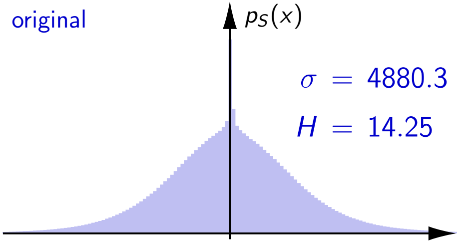
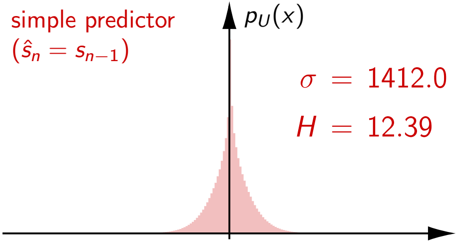
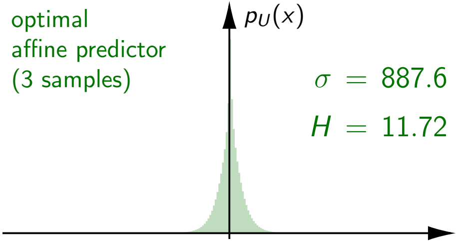
original



simple predictor
($\hat{s}_n = s_{n-1}$)



optimal
affine predictor
(3 samples)



Potential Improvements for Audio Coding

1. **Use more samples as observation set**
   - ➡ May improve the affine predictor
   - ➡ Requires transmission of more prediction parameters
   - ➡ Audio compression format **FLAC** uses up to 32 preceding samples

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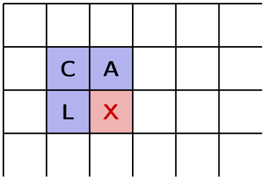
   - ➡ Requires transmission of more prediction parameters
   - ➡ Audio compression format **FLAC** uses up to 32 preceding samples

2. **Adapt predictor during coding**
   - Audio signals have instationary properties
   - ➡ A single predictor may not be suitable for all parts of an audio stream
   - ➡ Split data into chunks and determine best predictor for each chunk

Example: Prediction of Image Signals using Three Neighboring Samples

1 Choice of prediction structure

$$\hat{X} = a_0 + a_L \cdot L + a_A \cdot A + a_C \cdot C$$



Example: Prediction of Image Signals using Three Neighboring Samples

1 Choice of prediction structure

$$\hat{X} = a_0 + a_L \cdot L + a_A \cdot A + a_C \cdot C$$

2 Determine mean and required correlation factors

$$\varrho_{\text{hor}} = \frac{\sum_{x,y}(s[x,y]-\mu_S)(s[x-1,y]-\mu_S)}{\sum_{x,y}(s[x,y]-\mu_S)^2}$$

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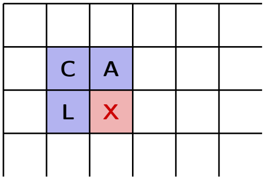
$$\varrho_{\text{al}} = \frac{\sum_{x,y}(s[x,y]-\mu_S)(s[x-1,y-1]-\mu_S)}{\sum_{x,y}(s[x,y]-\mu_S)^2}$$

$$\varrho_{\text{ar}} = \frac{\sum_{x,y}(s[x,y]-\mu_S)(s[x+1,y-1]-\mu_S)}{\sum_{x,y}(s[x,y]-\mu_S)^2}$$



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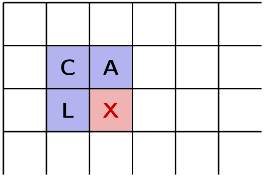

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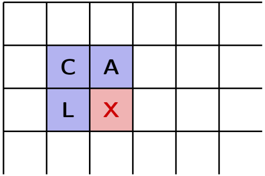
$$\begin{aligned} \mu_S &= 124.05 \\ \varrho_{\mathrm{hor}} &= 0.9722 \\ \varrho_{\mathrm{ver}} &= 0.9850 \\ \varrho_{\mathrm{al}} &= 0.9598 \\ \varrho_{\mathrm{ar}} &= 0.9689 \end{aligned}$$

Example: Prediction of Image Signals using Three Neighboring Samples

3 Solve linear equation system

$$\begin{bmatrix} 1 & \varrho_{\mathrm{ar}} & \varrho_{\mathrm{ver}} \\ \varrho_{\mathrm{ar}} & 1 & \varrho_{\mathrm{hor}} \\ \varrho_{\mathrm{ver}} & \varrho_{\mathrm{hor}} & 1 \end{bmatrix} \cdot \begin{bmatrix} a_L \\ a_A \\ a_C \end{bmatrix} = \begin{bmatrix} \varrho_{\mathrm{hor}} \\ \varrho_{\mathrm{ver}} \\ \varrho_{\mathrm{al}} \end{bmatrix} \quad \Longrightarrow$$



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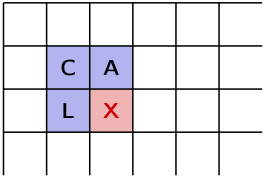

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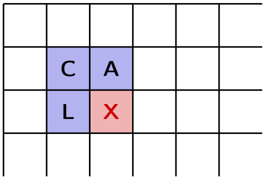

4 Determine constant offset

$$a_0 = \mu_S \left(1 - a_L - a_A - a_C\right) \quad \Longrightarrow \quad a_0 = 1.4198$$

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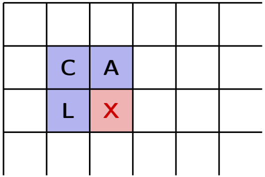

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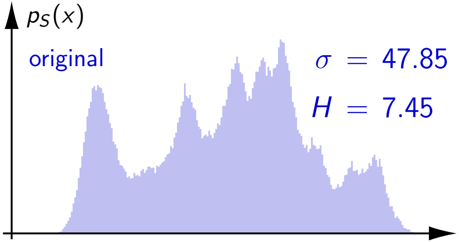
➡ Predictor is given by

$$\begin{aligned} \hat{X} &= \operatorname{round}\Big(a_0 + a_L \cdot L + a_A \cdot A + a_C \cdot C \Big) \\ &= \operatorname{round}\Big(1.4198 + 0.5892 \cdot L + 0.8255 \cdot A - 0.4262 \cdot C \Big) \end{aligned}$$

Example: Prediction of Image Signals using Three Neighboring Samples



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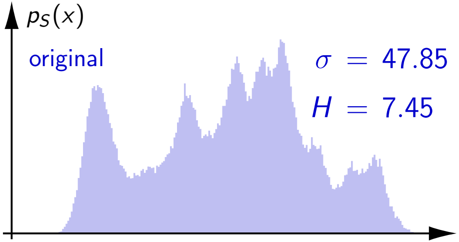

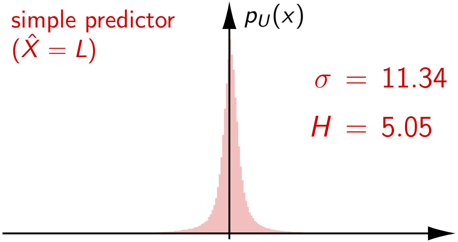

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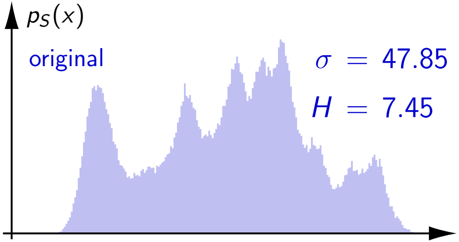

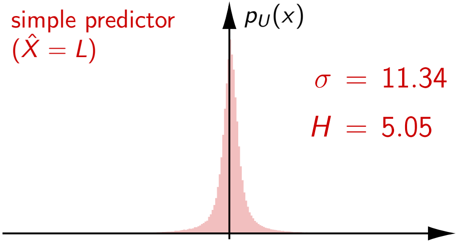

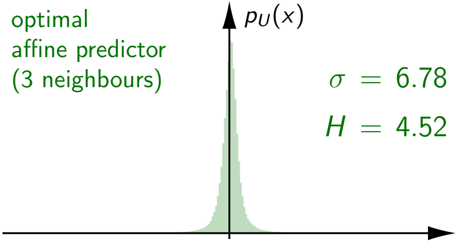

Potential Improvements for Image Coding

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2 Adapt predictor during coding

- Image signals have instationary properties: Direction of edges plays an important role

➡ Split image into blocks and choose predictor for each block

➡ May be sufficient to choose between pre-defined predictors

  - Horizontal predictor for parts with horizontal edges
  - Vertical predictor for parts with vertical edges
  - Some further predictors

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3 Non-linear Predictors

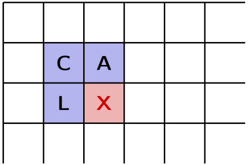
- ➡ Non-linear predictors may be able to deal with different edge directions

Examples for Non-Linear Predictors

LOCO Predictor used in JPEG-LS

- Each sample X is predicted according to

$$\hat{X} = \begin{cases} \min(L, A) & : \; C \geq \max(L, A) \\ \max(L, A) & : \; C \leq \min(L, A) \\ L + A - C & : \; \text{otherwise} \end{cases}$$



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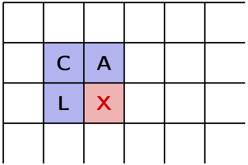
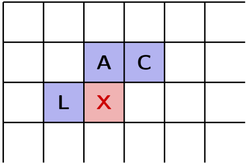


Motion Vector Prediction in Video Coding Standards

- Motion vector $\boldsymbol{m}$ is predicted by component-wise median

$$\hat{m}_x(X) \; = \; \mathrm{median}\Big(m_x(A), \; m_x(B), \; m_x(C) \Big)$$

$$\hat{m}_y(X) \; = \; \mathrm{median}\Big(m_y(A), \; m_y(B), \; m_y(C) \Big)$$



Summary of Lecture

Predictive Lossless Coding

- Entropy coding of prediction error signals $u_n = s_n - \hat{s}_n$
- Simple and effective way to exploit dependencies between neighbouring samples
- Complexity reduction relative to more general conditional entropy coding

Optimal Prediction

- Given by conditional mean for an observation set
- Complex due to requirement of large tables (similar to conditional entropy coding)

Affine and Linear Prediction

- Simple structure of predictor, low-complex implementations possible
- Optimal prediction parameters are given by solution of Yule-Walker equations
- For instationary sources (such as audio, image, video signals):
  - Determine predictor for smaller sets of samples (still large enough)
  - Determine optimal predictor or choose between set of predefined predictors

Exercise: Lossless Image Compression Challenge (Part I)

Implement an encoder and decoder for lossless coding of 8-bit color images:

1. We use the PPM format as raw data format:
    - The encoder should read the original images in PPM format.
    - The decoder should write the reconstructed images in PPM format.

   Example images (24 PPM images of the Kodak set) are provided on the course web-site (and in the KVV).

2. Use coding techniques that you learned for efficiently compressing the 8-bit color images.
    - A combination of prediction and entropy coding of the prediction errors is suggested.
    - Start with a simple (but working) approach and try to improve your codec step by step.

structure of "ppm" files:

```
P6                 // ascii (fixed)
width height       // ascii
255                // ascii (max. value)
<binary data>      // binary
```

binary data:

- pixels in raster-scan order (line by line)
- each pixel consists of three 8-bit values
    - ➡ R: red component (0..255)
    - ➡ G: green component (0..255)
    - ➡ B: blue component (0..255)
- the values R, G, B for a pixel follow each other (before the values for the next pixel)

suggestion:

- Store the red, green, and blue components of an image into separate arrays
- Code the color components independently