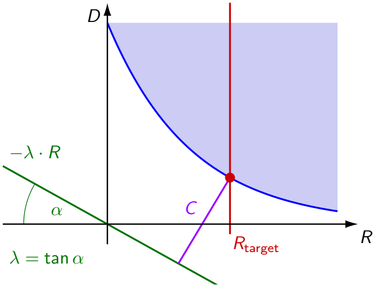
Optimal Scalar Quantization



Last Lectures: Lloyd Quantizer & Rate-Distortion Function

Lloyd Quantizer

- Minimizes distortion D for given number K of quantization intervals
- Design algorithm: Iterate between two optimization criterions
  - Centroid condition (MSE): $s'_k = \mathrm{E}\{\, S \,|\, S \in \mathcal{I}_k \,\}$
  - Nearest neighbor condition (MSE): $u_k = (s'_k + s'_{k-1})/2$

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Rate-Distortion Function

- Greatest lower bound for lossy source coding: $R(D) = \lim_{N\to\infty} \inf_{g_N:\ \delta_N(g_N)\le D} \dfrac{I_N(g_N)}{N}$
- Property of the source (no consideration of codes)

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High-Rate Approximations

- Panter & Dite asymptote for Lloyd quantizer (MSE): $D_F(R) = \varepsilon_F^2 \cdot \sigma^2 \cdot 2^{-2R}$
- Shannon lower bound for rate-distortion function (MSE): $D_L(R) = \varepsilon_L^2 \cdot \sigma^2 \cdot 2^{-2R}$

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➡ MSE distortion increase of Lloyd vs SLB: $\dfrac{D_F}{D_L}(R) = \dfrac{\varepsilon_F^2}{\varepsilon_L^2}$ ➡ Gauss: ≈ 2.72 (4.34 dB); Laplace: ≈ 5.20 (7.16 dB)

Rate-Distortion Efficiency of Scalar Quantizers

Distortion

- Quantifies deviation between original and reconstructed samples
- Typically: Additive distortion measures with $d(s, s')$ being the single-sample distortion

$$D = \mathrm{E}\{\, d(S, S')\,\} = \sum_{\forall k} \int_{u_k}^{u_{k+1}} d(s, s'_k)\, f(s)\, \mathrm{d}s$$

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- Average number of bits for coding quantization indexes q (or reconstructed values s')

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Design of Scalar Quantizers

- Lloyd quantizer minimizes distortion, but ignores impact on bit rate (assumes fixed-length coding)

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- Lloyd quantizer minimizes distortion, but ignores impact on bit rate (assumes fixed-length coding)

➡ Improved performance: **Consider bit rate in quantizer design**

Joint Minimization of Distortion and Bit Rate

Constrained Optimization Problem

- Optimization problem can be formulated as

$$\min \; D \qquad \text{subject to} \qquad R \le R_{\text{target}}$$

or, equivalently,

$$\min \; R \qquad \text{subject to} \qquad D \le D_{\text{target}}$$

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Reformulation as Unconstrained Problem

- Typically, constrained optimization problems cannot be solved directly
- Use technique of Lagrange multipliers for reformulation as unconstrained problem

$$\min \; D + \lambda \cdot R$$

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- Use technique of Lagrange multipliers for reformulation as unconstrained problem

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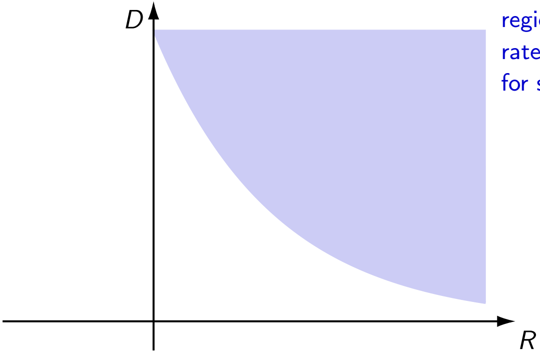
➡ The parameter $\lambda > 0$ is called **Lagrange multiplier**

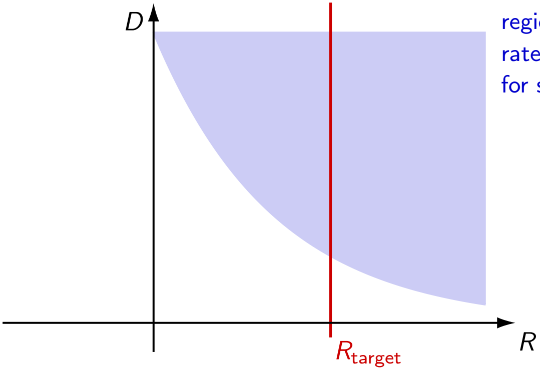
➡ Each value of λ corresponds to a rate constraint R_{target} (or distortion constraint D_{target})

Convex Optimization: Illustration



region of achievable rate-distortion points for scalar quantizer

Convex Optimization: Illustration



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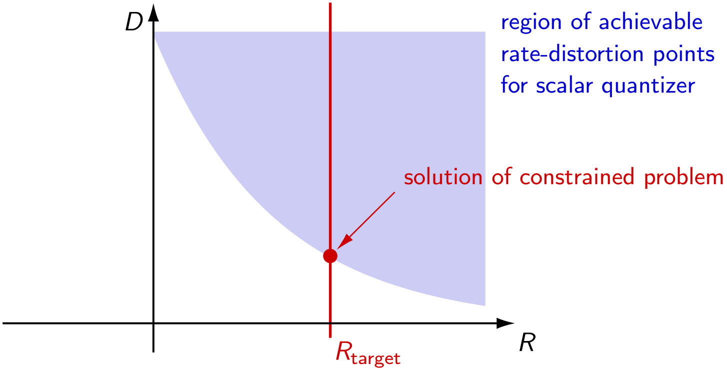

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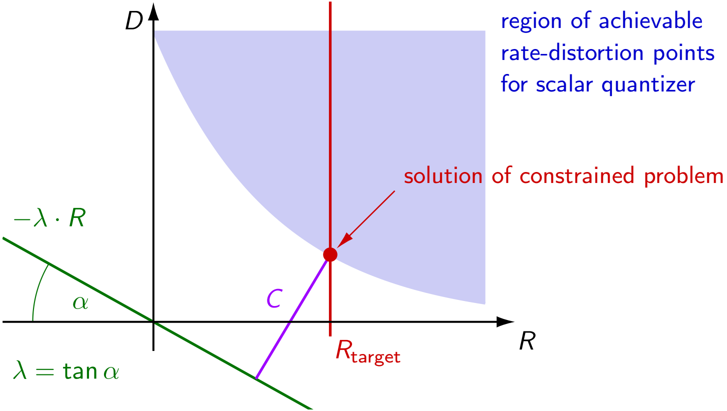

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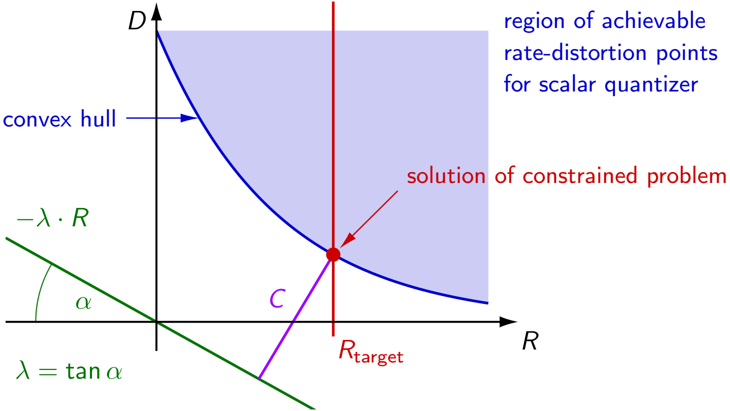

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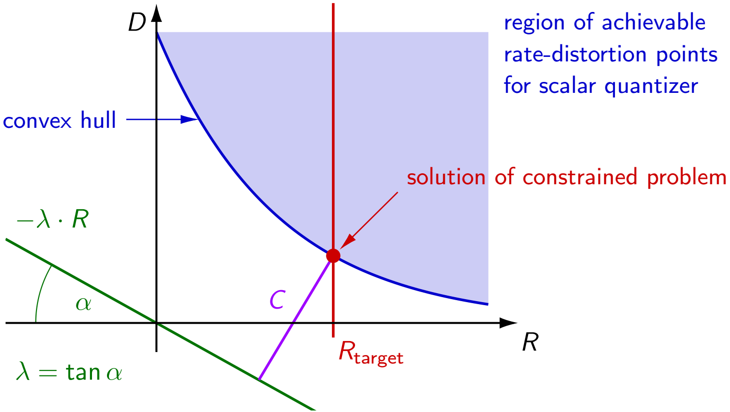
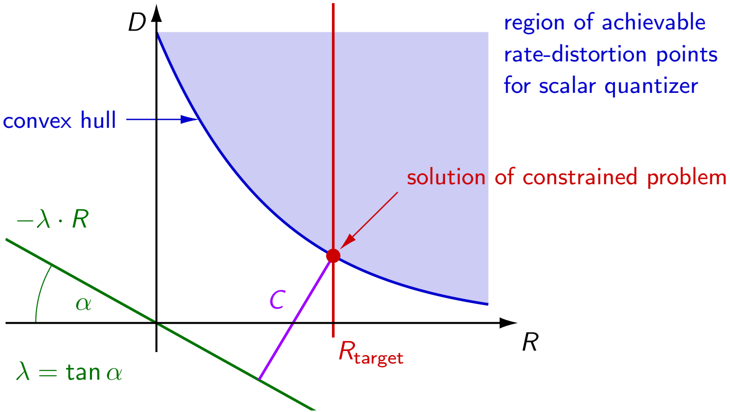


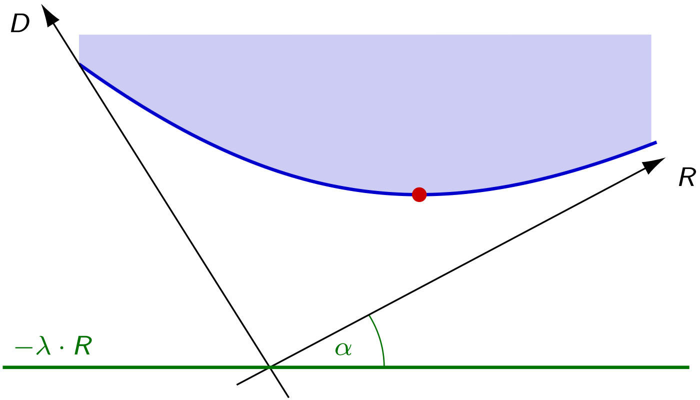
➡ Points on convex hull: Minimize distance C to line $D = -\lambda \cdot R$

Convex Optimization: Illustration



- ➡ Points on convex hull: Minimize distance C to line $D = -\lambda \cdot R$
- ➡ Geometrical interpretation: Rotate coordinate system by angle α

Convex Optimization: Lagrangian Formulation



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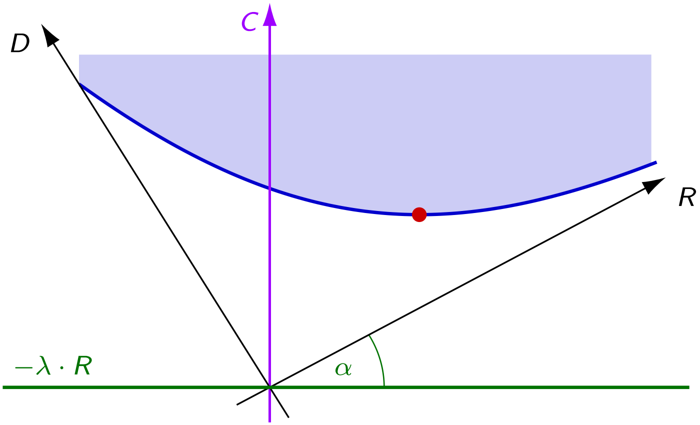

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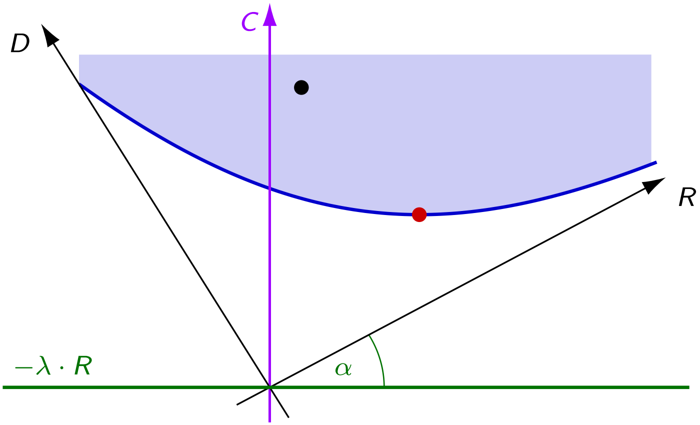

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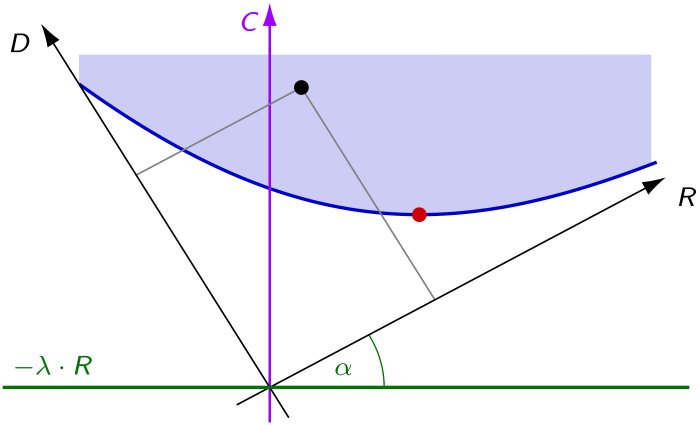

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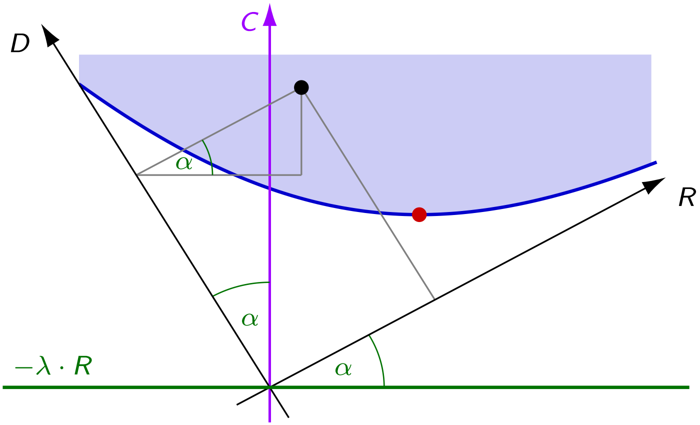

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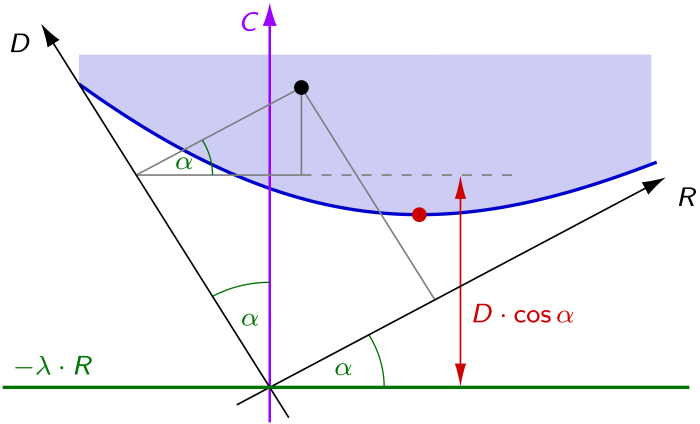

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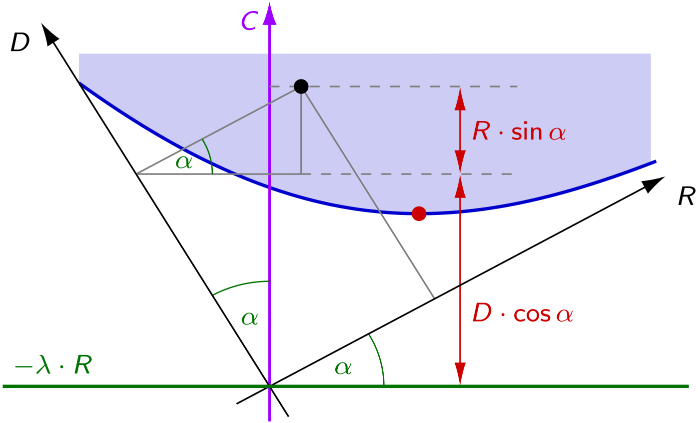

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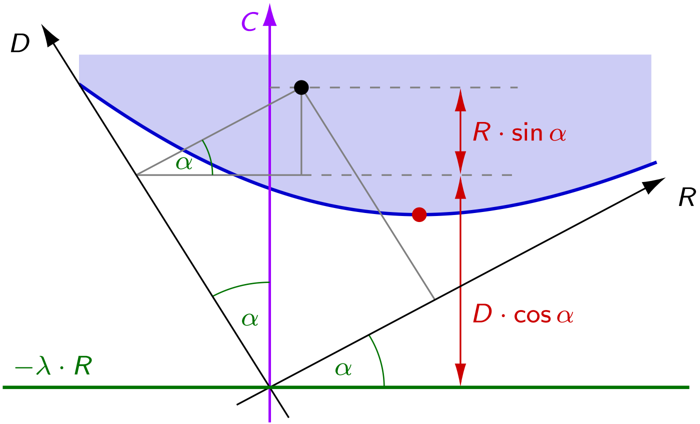

➡ Minimize distance: $C = D \cdot \cos\alpha + R \cdot \sin\alpha$

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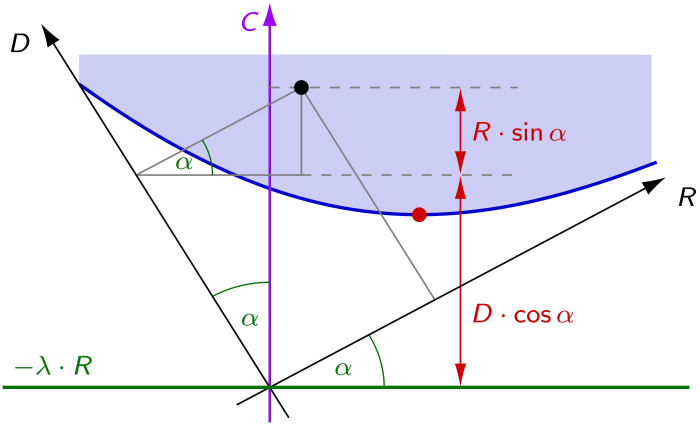

➡ Minimize distance: $C = D \cdot \cos\alpha + R \cdot \sin\alpha$

➡ Equivalent minimization: $J = D + \lambda \cdot R$ (note: Lagrange multiplier is given by $\lambda = \tan\alpha$)

Optimal Scalar Quantizer

Optimization Criterion

- Minimization of Lagrangian cost for some given Lagrange multiplier λ

$$J = D + \lambda \cdot R$$

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\begin{aligned}
J &= D + \lambda \cdot R \\
&= \sum_{\forall k} \int_{u_k}^{u_{k+1}} d(s, s'_k)\, f(s)\, \mathrm{d}s + \lambda \sum_{\forall k} \ell_k \int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s \qquad \text{(any additive distortion measure)}
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➡ Each Lagrange multiplier $\lambda > 0$ yields solution of original constrained problem for one R_{target}

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Optimization Criterion for MSE Distortion

- Determine quantizer parameters (s'_k and u_k) and codeword lengths ℓ_k such that the Lagrangian cost $J = D + \lambda R$ for MSE distortion is minimized

$$J = \sum_{\forall k} \int_{u_k}^{u_{k+1}} (s - s'_k)^2\, f(s)\, \mathrm{d}s + \lambda \sum_{\forall k} \ell_k \int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s$$

Optimal Decoder Mapping: Centroid Condition

$$J = \sum_{\forall k} \int_{u_k}^{u_{k+1}} (s - s'_k)^2 \, f(s) \, \mathrm{d}s \; + \; \lambda \cdot \sum_{\forall k} \ell_k \int_{u_k}^{u_{k+1}} f(s) \, \mathrm{d}s$$

1 **Optimize reconstruction levels s'_k** for given decision thresholds u_k

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➡ Centroid condition for MSE distortion

- Optimal reconstruction level s'_k is given by conditional mean

$$\boxed{s'_k = \mathrm{E}\{\, S \mid S \in \mathcal{I}_k \,\} = \int_{u_k}^{u_{k+1}} s\, f(s \,|\, s \in \mathcal{I}_k)\, \mathrm{d}s = \frac{1}{p_k} \int_{u_k}^{u_{k+1}} s\, f(s)\, \mathrm{d}s = \frac{\displaystyle\int_{u_k}^{u_{k+1}} s\, f(s)\, \mathrm{d}s}{\displaystyle\int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s}}$$

Optimal Lossless Mapping: Entropy Condition

$$J = \sum_{\forall k} \int_{u_k}^{u_{k+1}} (s - s'_k)^2 \, f(s) \, \mathrm{d}s \; + \; \lambda \cdot \sum_{\forall k} \ell_k \int_{u_k}^{u_{k+1}} f(s) \, \mathrm{d}s$$

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➡ Remember: Lossless coding theorem: $R \geq H(S')$

$$\sum_{\forall k} p_k \cdot \ell_k \geq -\sum_{\forall k} p_k \cdot \log_2 p_k \qquad \text{(equality if and only if } \ell_k = -\log_2 p_k\text{)}$$

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➡ **Entropy condition** (neglecting inefficiency of actual entropy coding)

$$\boxed{\ell_k = -\log_2 p_k = -\log_2 \left(\int_{u_k}^{u_{k+1}} f(s) \, \mathrm{d}s \right)}$$

Optimal Encoder Mapping: Condition for Decision Thresholds

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3 **Optimize decision thresholds** u_k for given reconstruction levels s'_k and codeword lengths ℓ_k

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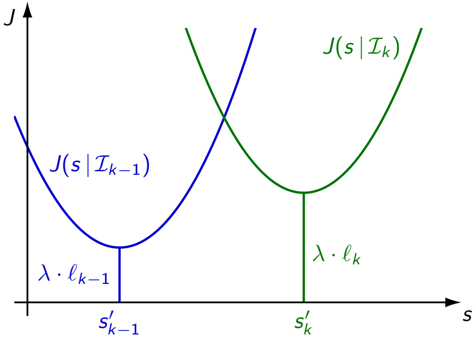
➡ Map each input value s to interval $\mathcal{I}_k$ that minimizes contribution to Lagrangian cost

$$J(s \,|\, \mathcal{I}_k) = (s - s'_k)^2 + \lambda \cdot \ell_k$$



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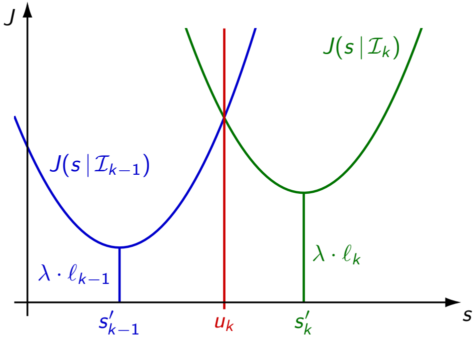
➡ At decision threshold u_k, we require

$$J(u_k \,|\, \mathcal{I}_{k-1}) = J(u_k \,|\, \mathcal{I}_k)$$



Optimal Encoder Mapping: Modified Nearest Neighbour Condition

- Optimal encoding mapping for MSE distortion

$$J(u_k \,|\, \mathcal{I}_{k-1}) \;=\; J(u_k \,|\, \mathcal{I}_k)$$

Optimal Encoder Mapping: Modified Nearest Neighbour Condition

- Optimal encoding mapping for MSE distortion

$$
\begin{aligned}
J(u_k \,|\, \mathcal{I}_{k-1}) &= J(u_k \,|\, \mathcal{I}_k) \\
(u_k - s'_{k-1})^2 + \lambda \cdot \ell_{k-1} &= (u_k - s'_k)^2 + \lambda \cdot \ell_k
\end{aligned}
$$

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2u_k(s'_k - s'_{k-1}) &= (s'_k)^2 - (s'_{k-1})^2 + \lambda\,(\ell_k - \ell_{k-1})
\end{aligned}
$$

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J(u_k \,|\, \mathcal{I}_{k-1}) &= J(u_k \,|\, \mathcal{I}_k) \\
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\end{aligned}
$$

Optimal Encoder Mapping: Modified Nearest Neighbour Condition

- Optimal encoding mapping for MSE distortion

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➜ **Modified nearest neighbour condition for MSE distortion**

$$
\boxed{u_k = \frac{1}{2}\left(s'_{k-1} + s'_k\right) + \frac{\lambda}{2}\left(\frac{\ell_k - \ell_{k-1}}{s'_k - s'_{k-1}}\right)}
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$$

➡ **Threshold is shifted towards the reconstruction level with the longer codeword**

Entropy-Constrained Lloyd Quantizer: Minimization of Lagrangian Cost

Necessary Conditions for Optimality (MSE distortion)

1. Centroid condition

$$s'_k = \frac{\int_{u_k}^{u_{k+1}} s\, f(s)\, \mathrm{d}s}{\int_{u_k}^{u_{k+1}} f(s)\, \mathrm{d}s}$$

2. Entropy condition

$$\ell_k = -\log_2 p_k$$

3. Modified nearest neighbor condition

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Design of optimal entropy-constrained scalar quantizers

- In general: Cannot be derived in closed form

➡ Iterative algorithm similar to Lloyd algorithm

Entropy-Constrained Lloyd Algorithm for Given Pdf (MSE Distortion)

Given is:

- the marginal probability density function $f(s)$ of the source
- a Lagrange multiplier $\lambda > 0$

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$$s'_k = \frac{\int_{u_k}^{u_{k+1}} s\, f(s)\,\mathrm{d}s}{\int_{u_k}^{u_{k+1}} f(s)\,\mathrm{d}s} \qquad \text{and} \qquad \ell_k = -\log_2\left(\int_{u_k}^{u_{k+1}} f(s)\,\mathrm{d}s\right)$$

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4. Repeat the previous two steps until convergence

Entropy-Constrained Lloyd Algorithm for a Training Set (MSE Distortion)

Given is:
- a sufficiently large realization $\{s_n\}$ of the considered source
- a Lagrange multiplier $\lambda > 0$

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- a sufficiently large realization $\{s_n\}$ of the considered source
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Iterative quantizer design

1. Choose an initial set of reconstruction levels $\{s'_k\}$ and codeword lengths $\{\ell_k\}$
2. Associate all samples of the training set $\{s_n\}$ with one of quantization intervals $\mathcal{I}_k$

$$q(s_n) = \arg\min_{\forall k} \; (s_n - s'_k)^2 + \lambda \cdot \ell_k$$

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3. Update the reconstruction levels $\{s'_k\}$ and codeword lengths $\{\ell_k\}$ according to

$$s'_k = \frac{1}{N_k} \sum_{n:\; q(s_n)=k} s_n \qquad \text{and} \qquad \ell_k = -\log_2\left(\frac{N_k}{N}\right)$$

where N_k is the number of samples associated with $\mathcal{I}_k$ and N is the total number of samples

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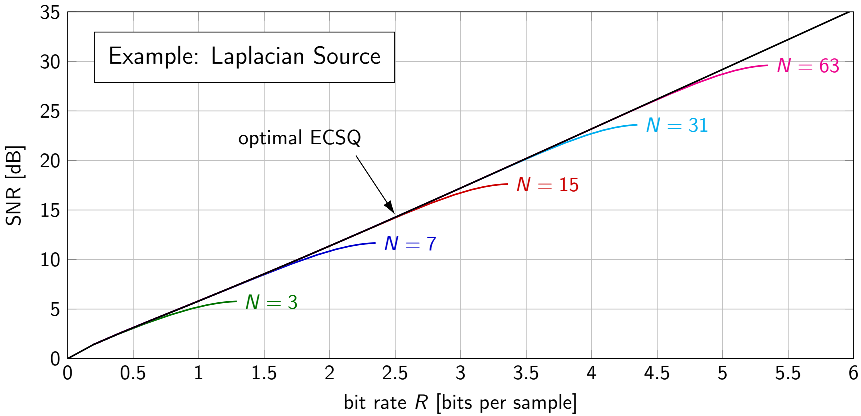
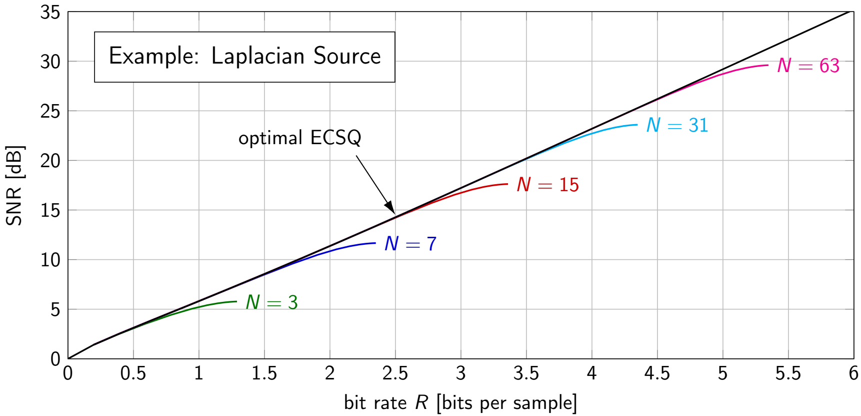
Number of Initial Intervals for Entropy-Constrained Lloyd Algorithm



- Too small number of intervals leads to sub-optimal design

Number of Initial Intervals for Entropy-Constrained Lloyd Algorithm



- Too small number of intervals leads to sub-optimal design
- EC Lloyd algorithm removes intervals during iterations (probabilities get smaller and smaller)

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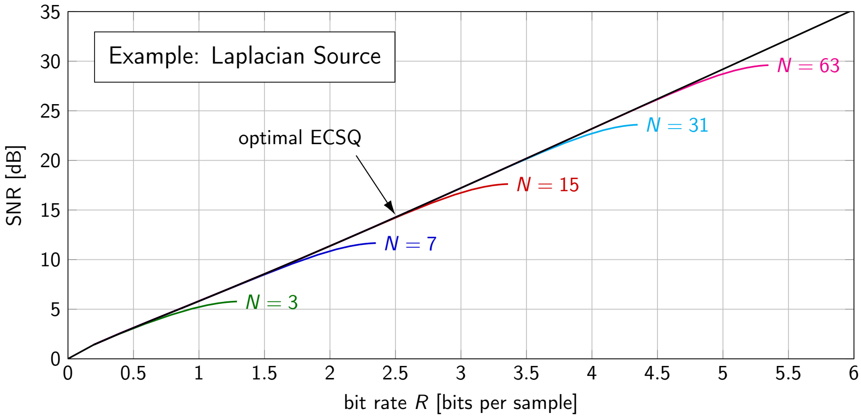


- Too small number of intervals leads to sub-optimal design
- EC Lloyd algorithm removes intervals during iterations (probabilities get smaller and smaller)

➜ **Use large number of intervals in initialization**

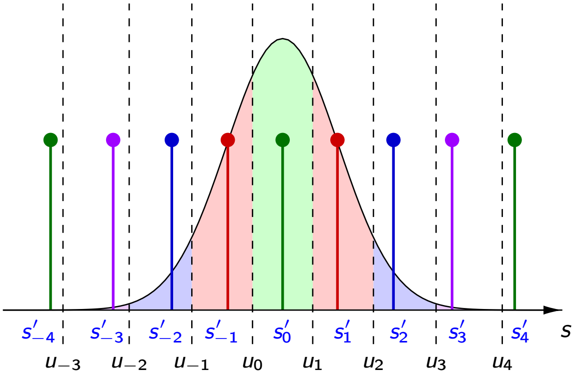
Example: EC Lloyd Algorithm for Gaussian Source

Gaussian Source

- Zero mean $\mu = 0$
- Unit variance $\sigma^2 = 1$

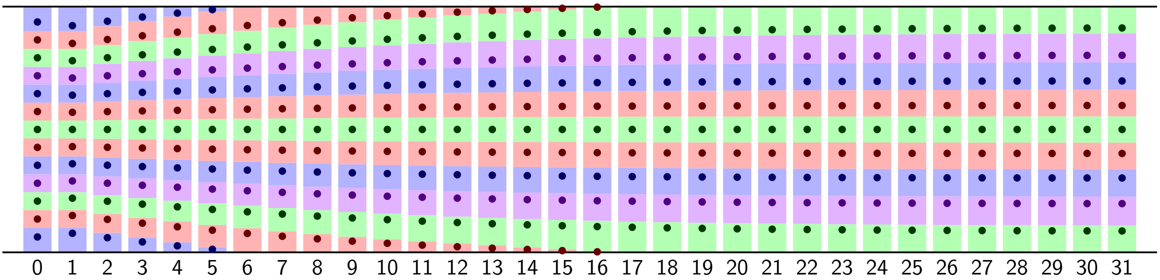
EC Lloyd Quantizer for 2 bits per sample

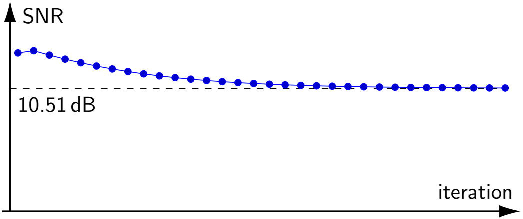
- Decision thresholds:
$$\begin{aligned} u_{0/1} &= \pm 0.538 \\ u_{-1/2} &= \pm 1.623 \\ u_{-2/3} &= \pm 2.743 \\ u_{-3/4} &= \pm 3.926 \end{aligned}$$
- Reconstruction levels:
$$\begin{aligned} s'_0 &= 0.000 \\ s'_{\pm 1} &= \pm 0.980 \\ s'_{\pm 2} &= \pm 1.981 \\ s'_{\pm 3} &= \pm 3.029 \\ s'_{\pm 4} &= \pm 4.148 \end{aligned}$$

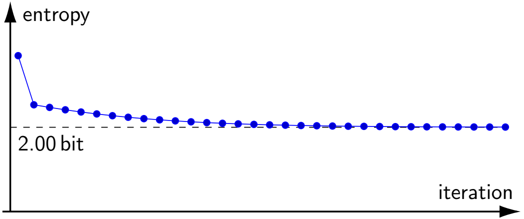


$$\begin{aligned} R &= 2.00 && \text{(rate = entropy)} \\ D &= 0.089 \\ \text{SNR} &= 10.51\,\text{dB} && \text{(RD bound} = 12.04\,\text{dB)} \end{aligned}$$

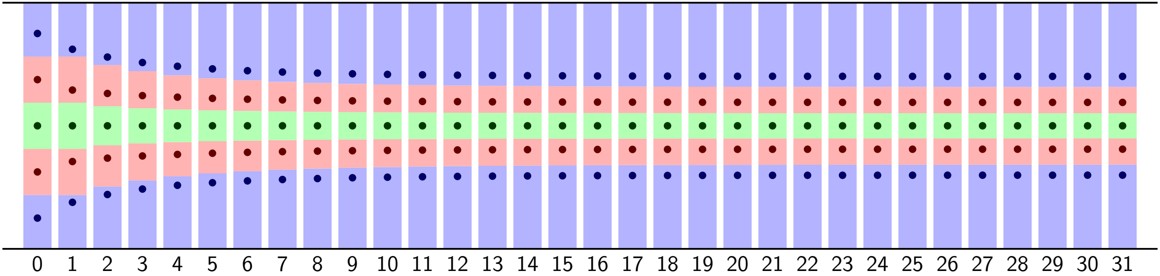
Example: Convergence of EC Lloyd Algorithm for Gaussian Source

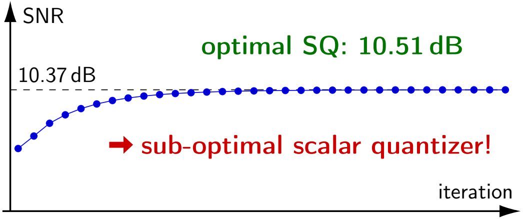


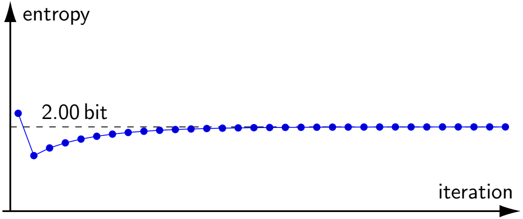




Example: EC Lloyd with Insufficient Initial Intervals (Gaussian Source)

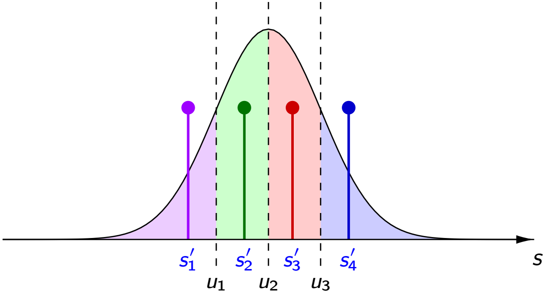






Example: EC Lloyd vs Lloyd at Same Entropy (Gaussian)

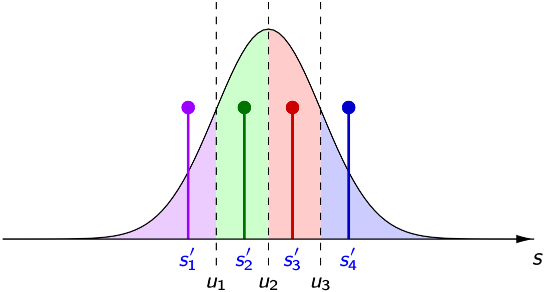
Lloyd Algorithm



$$
\begin{aligned}
K &= 4 \qquad\qquad (R_{\mathrm{FL}} = 2.0)\\
H &= 1.911\\
D &= 0.117\\
\mathrm{SNR} &= 9.30\ \mathrm{dB}
\end{aligned}
$$

Example: EC Lloyd vs Lloyd at Same Entropy (Gaussian)

Lloyd Algorithm



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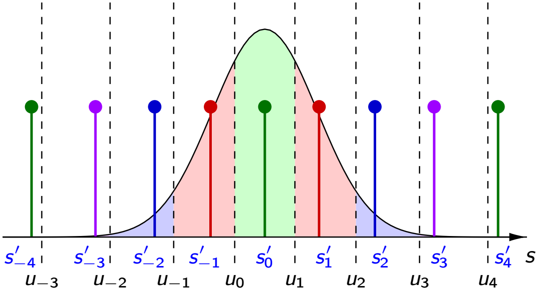
Entropy-Constrained Lloyd Algorithm



$$\lambda = 0.1393$$
$$H = 1.911$$
$$D = 0.101 \quad \Rightarrow \quad \text{factor 0.86 smaller}$$
$$\text{SNR} = 9.98\text{ dB} \quad \Rightarrow \quad 0.68\text{ dB better}$$

Example: EC Lloyd Algorithm for Laplacian Source

Laplacian Source

- Zero mean $\mu = 0$ and unit variance $\sigma^2 = 1$

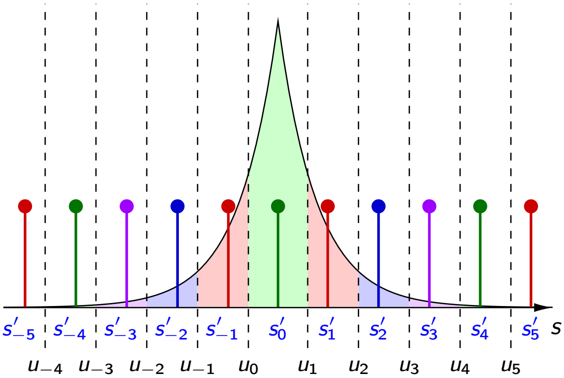
EC Lloyd Quantizer for 2 bits per sample

- Decision thresholds:

$$\begin{aligned} u_{0/1} &= \pm 0.540 \\ u_{-1/2} &= \pm 1.465 \\ u_{-2/3} &= \pm 2.390 \\ u_{-3/4} &= \pm 3.315 \\ u_{-4/5} &= \pm 4.240 \\ &\cdots \end{aligned}$$

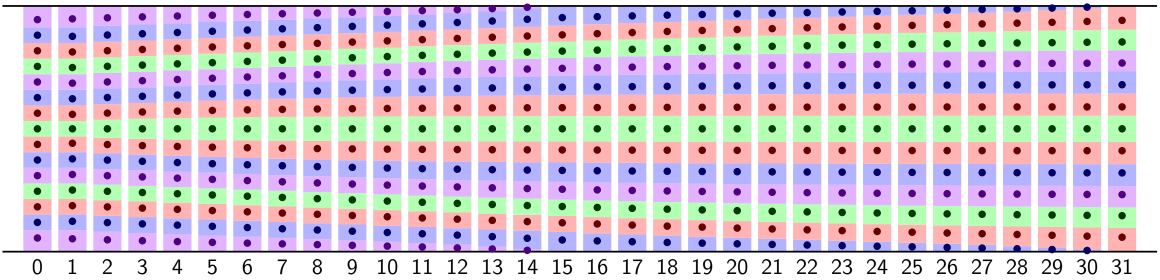
- Reconstruction levels:

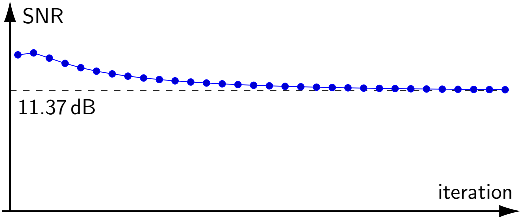
$$\begin{aligned} s'_0 &= 0.000 \\ s'_{\pm 1} &= \pm 0.905 \\ s'_{\pm 2} &= \pm 1.830 \\ s'_{\pm 3} &= \pm 2.755 \\ s'_{\pm 4} &= \pm 3.681 \\ s'_{\pm 5} &= \pm 4.606 \\ &\cdots \end{aligned}$$

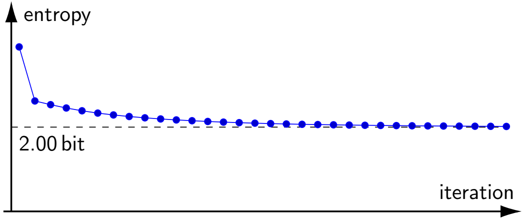


$$\begin{aligned} R &= 2.00 & &(\text{rate} = \text{entropy}) \\ D &= 0.073 \\ \text{SNR} &= 11.37\,\text{dB} & &(\text{SLB} = 12.67\,\text{dB}) \end{aligned}$$

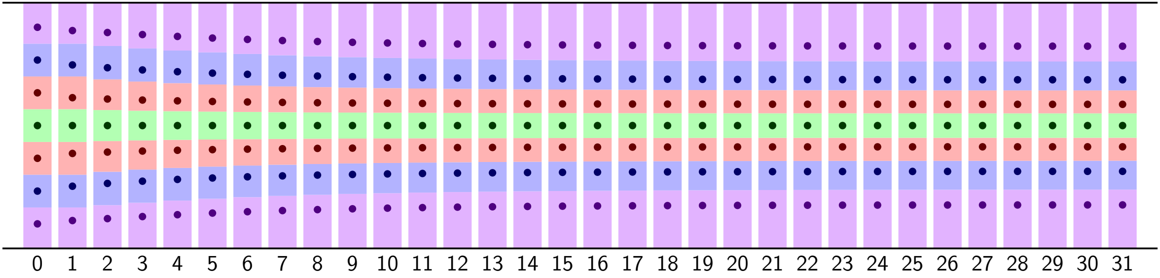
Example: Convergence of EC Lloyd Algorithm for Laplacian Source

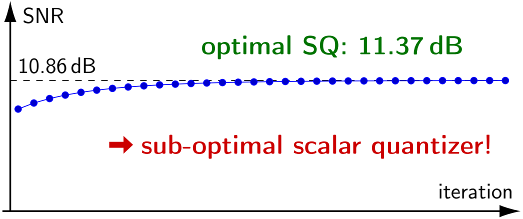


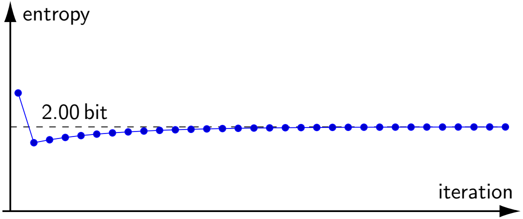




Example: EC Lloyd with Insufficient Initial Intervals (Laplacian Source)

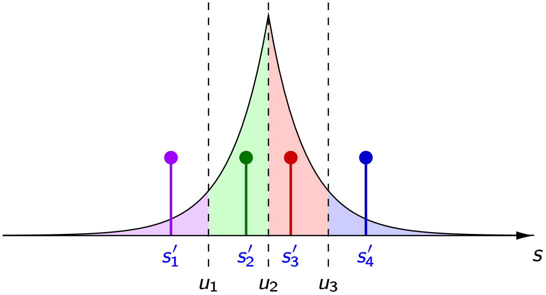






Example: EC Lloyd vs Lloyd at Same Entropy (Laplace)

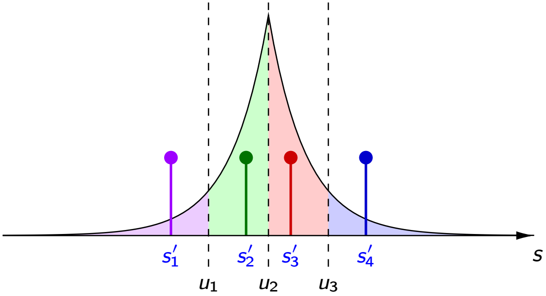
Lloyd Algorithm



$$\begin{aligned} K &= 4 \qquad\qquad (R_{\mathrm{FL}} = 2.0) \\ H &= 1.728 \\ D &= 0.176 \\ \mathrm{SNR} &= 7.54\ \mathrm{dB} \end{aligned}$$

Example: EC Lloyd vs Lloyd at Same Entropy (Laplace)

Lloyd Algorithm



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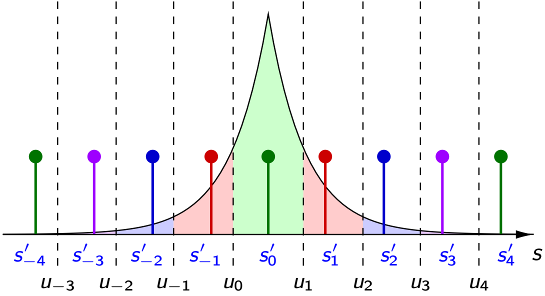
Entropy-Constrained Lloyd Algorithm



$$
\begin{aligned}
\lambda &= 0.1350\\
H &= 1.728\\
D &= 0.104 \quad \Rightarrow \quad \text{factor 0.59 smaller}\\
\text{SNR} &= 9.83\text{ dB} \quad \Rightarrow \quad \text{2.29 dB better}
\end{aligned}
$$

Review: MSE Distortion for Centroid Quantizers at High Rates

High-Rate Approximation

- High rates: Pdf $f(s)$ is nearly constant inside each quantization interval

$$f(s) \approx \frac{p_k}{\Delta_k} = \frac{p_k}{u_{k+1} - u_k} \qquad \Longrightarrow \qquad p_k \approx f(s'_k) \cdot \Delta_k$$

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MSE Distortion for Centroid Quantizers at High Rates

- When considering Lloyd quantizers, we derived

$$\boxed{D = \frac{1}{12} \sum_{\forall k} p_k \, \Delta_k^2}$$

Entropy of Quantization Indexes at High Rates

Average Bit Rate for Optimal Entropy Coding

- Approximation for high bit rates ($\Delta_k \to 0$)

$$R \;=\; H(S') = -\sum_{\forall k} p_k \cdot \log_2 p_k$$

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R &= H(S') = -\sum_{\forall k} p_k \cdot \log_2 p_k \\
\big[\, p_k = f(s'_k)\,\Delta_k \,\big] \qquad &= -\sum_{\forall k} p_k \left(\log_2 f(s'_k) + \log_2 \Delta_k \right) \\
&= -\sum_{\forall k} f(s'_k)\, \log_2 f(s'_k)\, \Delta_k - \sum_{\forall k} p_k \log_2 \Delta_k
\end{aligned}
$$

Entropy of Quantization Indexes at High Rates

Average Bit Rate for Optimal Entropy Coding

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$\big[\, p_k = f(s'_k)\,\Delta_k \,\big]$

$$= -\sum_{\forall k} p_k \left(\log_2 f(s'_k) + \log_2 \Delta_k \right)$$

$$= -\sum_{\forall k} f(s'_k)\, \log_2 f(s'_k)\,\Delta_k - \sum_{\forall k} p_k \log_2 \Delta_k$$

$\big[\, \Delta_k \to 0 \,\big]$

Entropy of Quantization Indexes at High Rates

Average Bit Rate for Optimal Entropy Coding

- Approximation for high bit rates ($\Delta_k \rightarrow 0$)

$$
\begin{aligned}
R &= H(S') = -\sum_{\forall k} p_k \cdot \log_2 p_k \\
\left[\, p_k = f(s_k')\,\Delta_k \,\right] \qquad &= -\sum_{\forall k} p_k \left(\log_2 f(s_k') + \log_2 \Delta_k \right) \\
&= -\sum_{\forall k} f(s_k')\, \log_2 f(s_k')\, \Delta_k - \sum_{\forall k} p_k \log_2 \Delta_k \\
\left[\, \Delta_k \rightarrow 0 \,\right] \qquad &= -\int_{-\infty}^{\infty} f(s)\, \log_2 f(s)\, \mathrm{d}s - \frac{1}{2} \sum_{\forall k} p_k \log_2 \Delta_k^2
\end{aligned}
$$

Entropy of Quantization Indexes at High Rates

Average Bit Rate for Optimal Entropy Coding

- Approximation for high bit rates ($\Delta_k \to 0$)

$$
\begin{aligned}
R &= H(S') = -\sum_{\forall k} p_k \cdot \log_2 p_k \\
\big[\, p_k = f(s'_k)\,\Delta_k \,\big] \qquad &= -\sum_{\forall k} p_k \Big(\log_2 f(s'_k) + \log_2 \Delta_k \Big) \\
&= -\sum_{\forall k} f(s'_k)\, \log_2 f(s'_k)\, \Delta_k - \sum_{\forall k} p_k \log_2 \Delta_k \\
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\end{aligned}
$$

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High-Rate Approximation: Optimal Entropy-Constrained Scalar Quantizers

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High-Rate Approximation for MSE Distortion: Gish & Pierce Asymptote

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- High-rate approximations for MSE distortion

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➡ **roughly 1/4 bit per sample**

Summary: High-Rate Approximations for MSE Distortion

- General form of high-rate approximations for MSE distortion

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general :	$\varepsilon_L^2 = \frac{1}{2\pi e}\, 2^{2h(S/\sigma)}$	$\varepsilon_V^2 = \frac{1}{12}\, 2^{2h(S/\sigma)}$	$\varepsilon_F^2 = \frac{1}{12}\int_{-\infty}^{\infty} \sqrt[3]{f(s/\sigma)}\,\mathrm{d}s$

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Comparison of Quantizers and High-Rate Approximations: Gaussian Source

High-rate approximations

$$D_X(R) = \varepsilon_X^2 \cdot \sigma^2 \cdot 2^{-2R}$$

SLB: $\varepsilon_L^2 = 1$

Lloyd: $\varepsilon_F^2 = \frac{\sqrt{3}\,\pi}{2}$

EC-Lloyd: $\varepsilon_V^2 = \frac{\pi\, e}{6}$

$$\frac{D_F}{D_L} = \frac{\sqrt{3}\,\pi}{2} \approx 2.72 \quad (4.34\,\text{dB})$$

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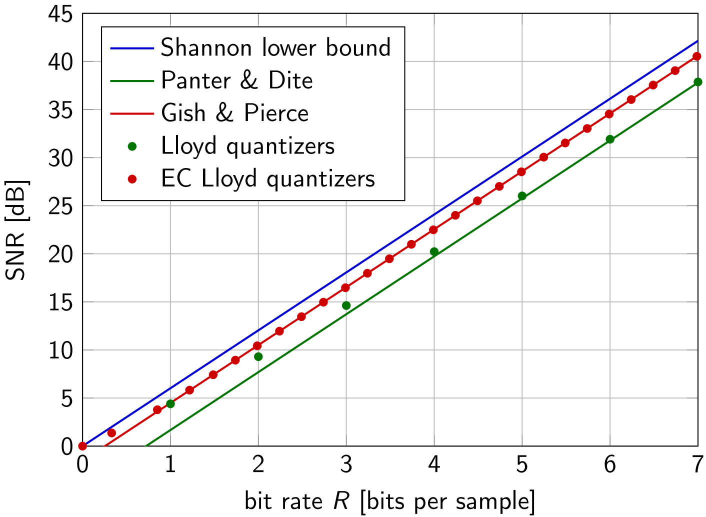

Comparison of Quantizers and High-Rate Approximations: Laplacian Source

High-rate approximations

$$D_X(R) = \varepsilon_X^2 \cdot \sigma^2 \cdot 2^{-2R}$$

SLB: $\varepsilon_L^2 = \dfrac{e}{\pi}$

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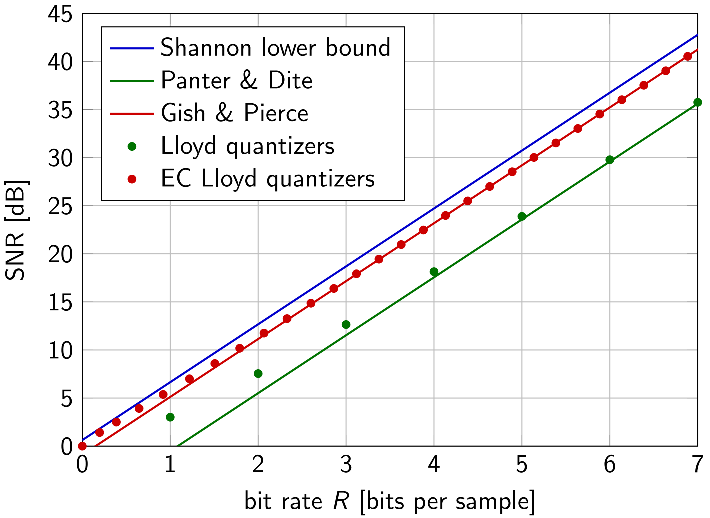

Scalar Quantization in Practice

Quantization in Practice

- Most quantizers used in practice are scalar quantizers
- Examples for usage of scalar quantization (in combination with other techniques):
  - Audio coding: MP3, AAC
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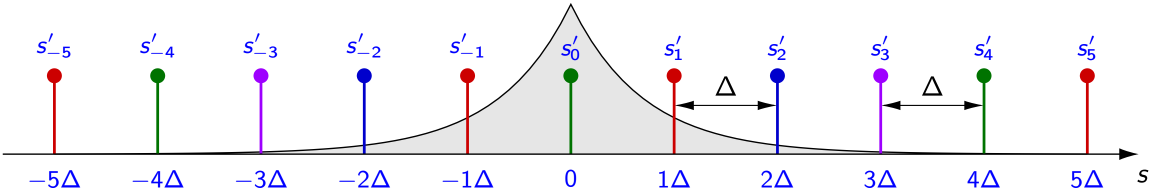
➡ **Need simpler, but still efficient design**

Uniform Reconstruction Quantizers (URQs)



Uniform reconstruction quantizers

- Equally spaced reconstruction levels (indicated by quantization step size Δ)

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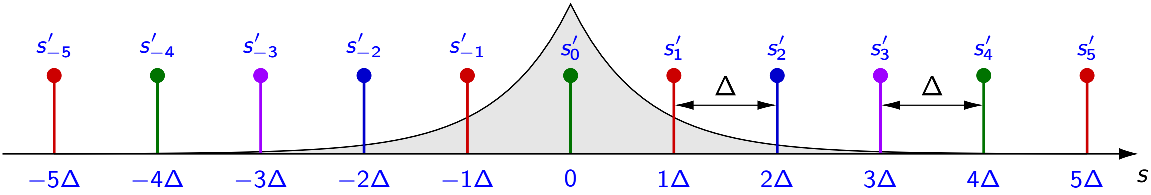

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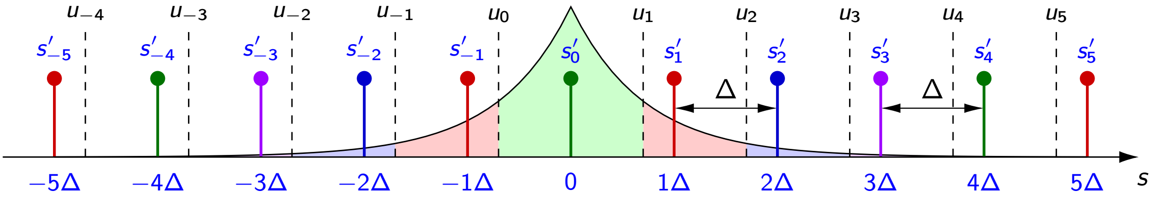

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➡ **Encoder:**
- Freedom to adapt decision thresholds to source statistics
- Simple encoding (rounding) or advanced encoding (Lagrange optimization)

Optimum Uniform Reconstruction Quantizer (URQ)

Optimum URQ Design for MSE Distortion

- Minimization of Lagrange cost for given Lagrange multiplier λ

$$J = D + \lambda \cdot R$$

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➡ Minimize J with respect to

- Decision thresholds u_k
- Codeword lengths ℓ_k
- Quantization step size Δ

Optimization Criterions for URQs with MSE Distortion

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$$u_k = \Delta \left(k - \frac{1}{2} \right) + \frac{\lambda}{2\Delta} \big(\ell_k - \ell_{k-1} \big) \qquad \Big[\text{note: } s'_k = k\Delta \Big]$$

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$$u_k = \Delta \left(k - \frac{1}{2} \right) + \frac{\lambda}{2\Delta} \left(\ell_k - \ell_{k-1} \right) \qquad \left[\text{note: } s'_k = k\Delta \right]$$

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Optimization Criterions for URQs with MSE Distortion

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Given is:

- the marginal probability density function $f(s)$ of the source
- a Lagrange multiplier $\lambda > 0$

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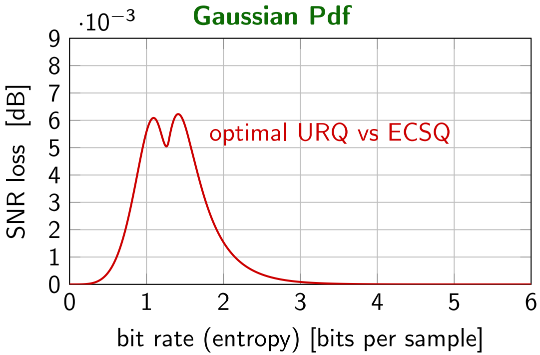
Note: Similar iterative algorithm for training set (instead of pdf)

Coding Efficiency Comparison: Optimal URQs vs ECSQs



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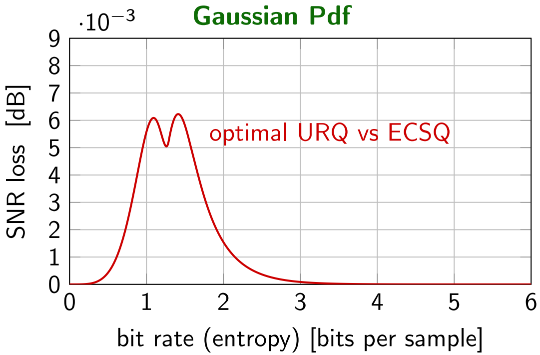

$$\Delta\text{SNR} < 0.0063 \quad \Rightarrow \quad \frac{D_{\text{URQ}}}{D_{\text{opt}}} < 1.0015$$

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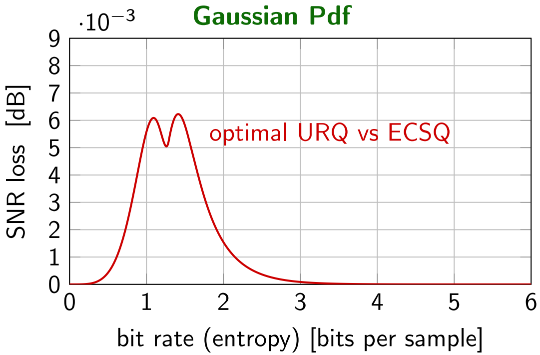

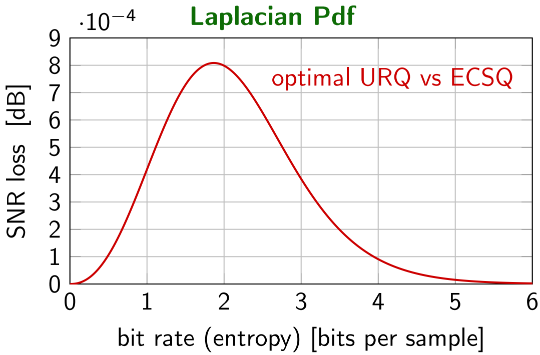

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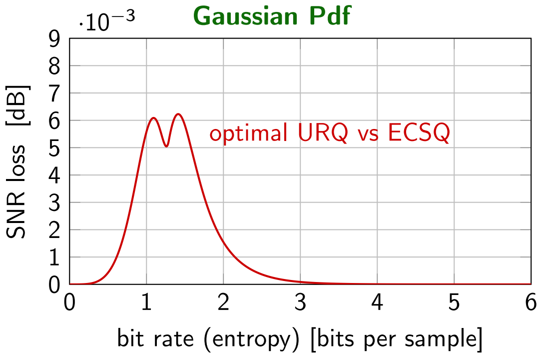

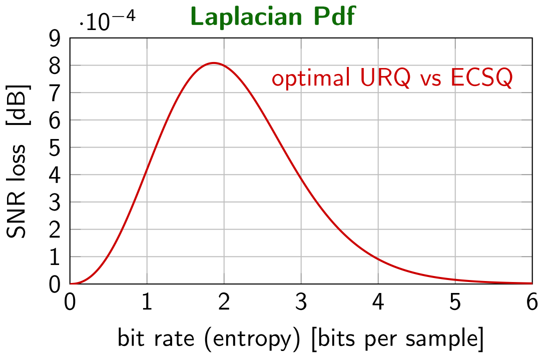

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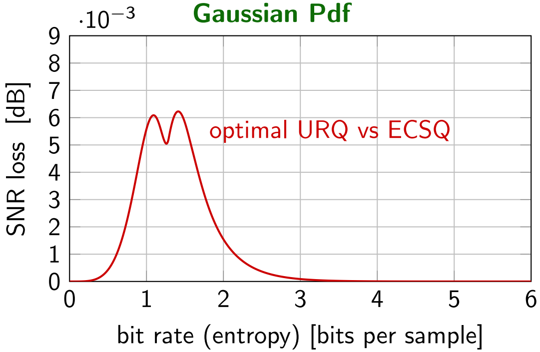

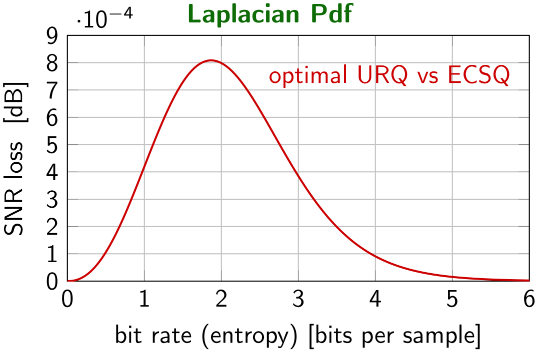

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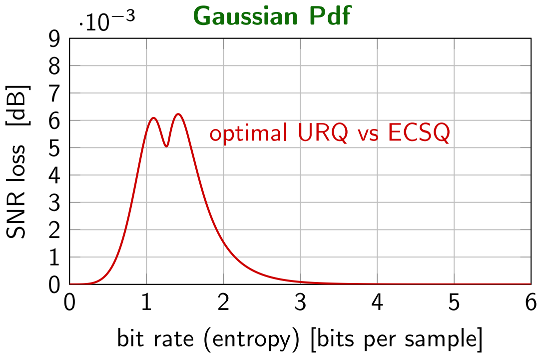

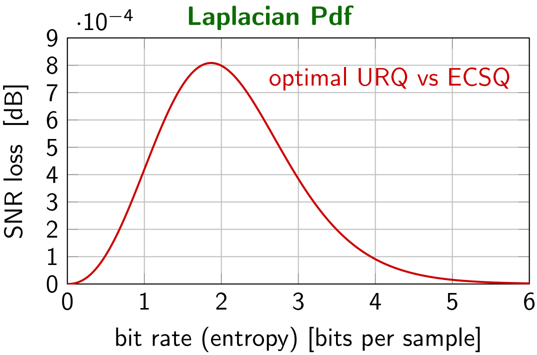

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➡ **For typical pdfs: Negligible loss versus optimal ECSQ**

➡ **Same high-rate performance as optimal ECSQ**

Quantization Step Size vs Lagrange Multiplier

- High-rate distortion approximations

$$D(R) = \varepsilon^2 \cdot \sigma^2 \cdot 2^{-2R}, \qquad D = \frac{1}{12} \sum_k p_k \, \Delta_k^2 = \frac{\Delta^2}{12} \qquad \text{(also valid for URQ)}$$

Quantization Step Size vs Lagrange Multiplier

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➡ **Often used relationship between λ and Δ**

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➡ Note: We only need to check the two neighboring reconstruction levels

$$k_1 = \lfloor s/\Delta \rfloor \qquad \text{and} \qquad k_2 = \lceil s/\Delta \rceil$$

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➡ Most quantizers used in practice are URQs

Summary of Lecture

Optimal Scalar Quantizers

- Minimizes Lagrangian cost $J = D + \lambda\, R$ (where $\lambda > 0$ determines operation point)
- Three optimization criterions: ➡ centroid condition
  ➡ entropy condition
  ➡ modified nearest neighbour condition
- May need large number ($K \to \infty$) of intervals for obtaining optimal quantizer
- High-rate approximation: ➡ distortion is factor 1.42 higher than SLB (1.53 dB)
  ➡ bit rate is roughly 0.25 bits per sample larger than SLB

Uniform Reconstruction Quantizers

- Uniformly spaced reconstruction levels (specified by quantization step size Δ)
- Very simple decoder mapping: $s' = \Delta \cdot k$
- Coding efficiency very close to optimal scalar quantizers (with suitable encoder decisions)

➡ **Most often used quantizer in practice**

Exercise 1: Implement the Entropy-Constrained Lloyd Algorithm (optional)

Implement the entropy-constrained Lloyd algorithm using a programming language of your choice.

- Test the algorithm for
  - a unit-variance Gaussian pdf:
$$f(s) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\, s^2}$$
  - a unit-variance Laplacian pdf:
$$f(s) = \frac{1}{\sqrt{2}}\, e^{-\sqrt{2}\,|s|}$$
- Use the following Lagrange multipliers: $\lambda = 0.5, 0.2, 0.1, 0.05, 0.02, 0.01$.
- Determine the rate (entropy) R and the distortion D for your quantizers.
- Compare the R-D performance of your quantizers to the high-rate approximation.

You can implement the EC Lloyd algorithm that directly uses the pdf or the EC Lloyd algorithm that uses a training set (files with 1 000 000 samples in float32 format are provided on the course web site)

Exercise 2: Quantization of Sources with Memory

Consider a discrete Markov process $\mathbf{X} = \{X_n\}$ with the symbol alphabet $\mathcal{A}_X = \{0, 2, 4, 6\}$ and the conditional pmf

$$p_{X_n|X_{n-1}}(x_n|x_{n-1}) = \begin{cases} a & : \quad x_n = x_{n-1} \\ \frac{1}{3}(1-a) & : \quad x_n \neq x_{n-1} \end{cases}$$

The parameter a, with $0 < a < 1$, is a variable that specifies the probability that the current symbol is equal to the previous symbol. For $a = 1/4$, our source $\mathbf{X}$ would be i.i.d.

Given is a two-interval quantizer with the reconstruction levels $s'_0 = 1$ and $s'_1 = 5$ and the decision threshold $u_1 = 3$.

(a) Assume optimal entropy coding using the marginal probabilities of the quantization indices and determine the rate-distortion point of the quantizer.

(b) Can the overall quantizer performance be improved by applying conditional entropy coding (e.g., using arithmetic coding with conditional probabilities)?

How does it depend on the parameter a?

Exercise 3: High-Rate Quantization

Consider scalar quantization of a Laplacian source at high rates:

$$f(x) = \frac{\lambda}{2} \cdot e^{-\lambda\,|x|} \qquad \text{with} \qquad \sigma_S^2 = \frac{2}{\lambda^2}$$

In a given system, the used quantizer is a Lloyd quantizer with fixed-length entropy coding (the number of quantization intervals represents a power of 2).

How many bits per sample (for the same MSE distortion) can be saved if the quantizer is replaced by an entropy-constrained quantizer with optimal entropy coding?

Note:

Assume that the operation points of the quantizers can be accurately described by the corresponding high rate approximations.

Exercise 4: Implementation of First Lossy Image Codec

- Use the PPM format as raw data format (see earlier exercise on lossless image coding)
- Use any of the lossless image codecs available in the KVV (or your own implementation) as basis

Implement an Image Encoder

- Quantize the original image samples $s[x, y]$ using a fixed quantization step size Δ
  - ➡ Simple rounding is sufficient for our purpose: $k[x, y] = \mathrm{round}(\, s[x, y]/\Delta \,)$
  - ➡ Transmit the quantization step size Δ at the beginning of the bitstream
- Use the lossless codec for coding the quantization indexes $k[x, y]$

Implement the corresponding Image Decoder

- Decode the quantization indexes $k[x, y]$ using the lossless codec
- Reconstruct the image samples according to: $s'[x, y] = k[x, y] \cdot \Delta$

Test your Codec

- Code selected test images with different quantization step sizes (e.g., $\Delta = 2, 4, 8, 16, 32, 64$)
- Measure the compression factors (based on the file sizes) and judge the image quality by visual inspection

Exercise 5: Quantization of Exponential Source (optional / more difficult)

Consider uniform threshold quantization of an exponential pdf given by $f(x) = a\,e^{-ax}$.

With Δ denoting the quantization step size, the thresholds are given by $u_k = k\Delta$, with $k = 0, 1, 2, \cdots$.

(a) Determine the pmf for the quantization indexes.

Calculate the rate (entropy) as function of the probability $p = P(X > \Delta) = e^{-a\Delta}$.

Describe an entropy coding scheme for the quantization indices that virtually achieves the entropy.

(b) Derive a formula for the optimal reconstruction levels s'_k, for MSE distortion, as function of the quantization step size Δ, the lower interval boundaries u_k, and the probability $p = e^{-a\Delta}$.

(c) Is the obtained quantizer an optimal entropy-constrained scalar quantizer?

(d) Determine the distortion in dependence of the quantization step size for the developed quantizer.

Hint: For $|a| < 1$, $\displaystyle\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$, $\displaystyle\sum_{k=0}^{\infty} k\,a^k = \frac{a}{(1-a)^2}$, $\displaystyle\sum_{k=0}^{\infty} k^2\,a^k = \frac{a\,(1+a)}{(1-a)^3}$