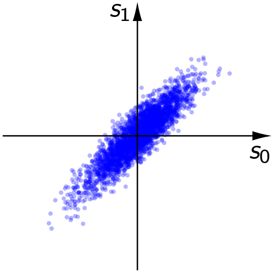
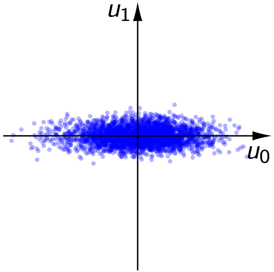
Transform Coding



$$\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$



Last Lectures: Scalar and Vector Quantization

Scalar Quantization

- Simple encoding and decoding procedure
- Uniform reconstruction quantizers (URQ): Particularly simple and still very efficient
- Cannot exploit statistical dependencies (would require very complex entropy coding)

Vector Quantization

- High-dimensional vector quantizers can approach rate-distortion bound
- Space-filling gain can only be exploited by vector quantization (1.53 dB for $N \rightarrow \infty$)
- Rarely used in practice: High computational complexity and memory requirements

Lossy Coding of Sources with Memory

- Most important aspect: Exploit statistical dependencies (memory advantage)
- Need approach that is simpler than vector quantization, but still efficient

Transform Coding: Introduction

Transform Coding

- Simple concept for exploiting linear dependencies between samples
- Low complexity compared to vector quantization (can be interpreted as very simple VQ)
- **Used in virtually all lossy audio, image and video codecs**

Basic Concept

1. Arrange samples into blocks/vectors $\boldsymbol{s}$ of N adjacent samples
2. **Analysis transform**: Mapping to vectors of transform coefficients

$$\boldsymbol{u} = \boldsymbol{A} \cdot \boldsymbol{s}$$

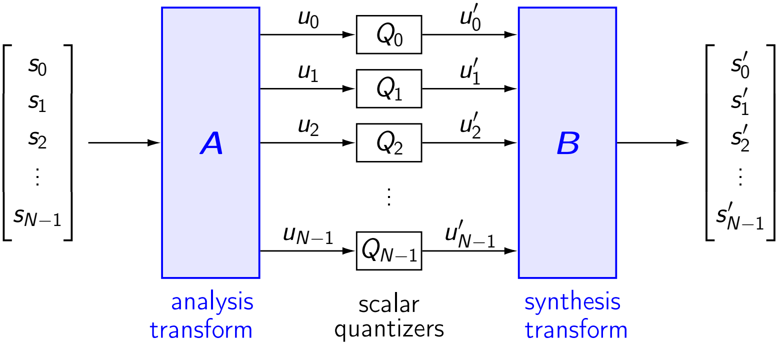
3. **Scalar quantization** of transform coefficients $\boldsymbol{u} = \{u_k\}$

$$u_k \mapsto u'_k$$

4. **Synthesis transform**: Mapping to blocks/vectors of reconstructed samples

$$\boldsymbol{s}' = \boldsymbol{B} \cdot \boldsymbol{u}'$$

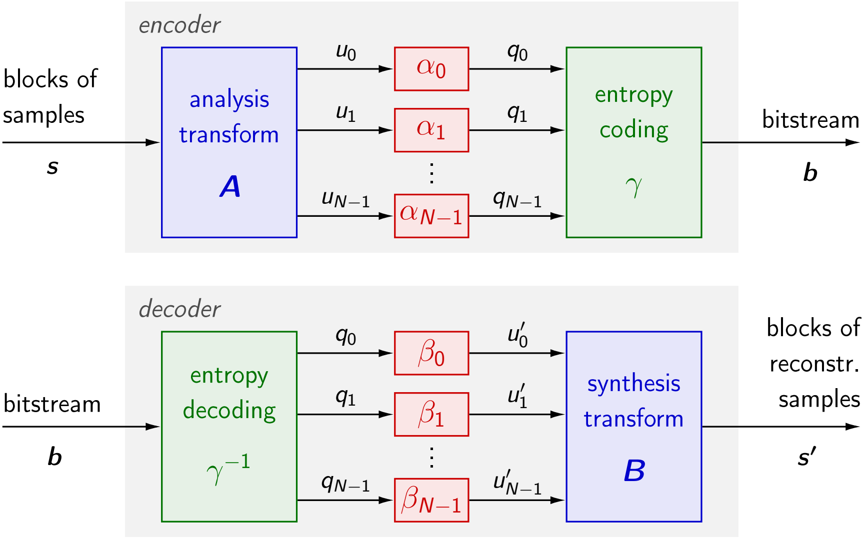
Structure of Transform Coding Systems



Effect of transform coding:

- Remove/reduce dependencies before scalar quantization
- Simple alternative to vector quantization
- Simple and most relevant case: Linear transforms

Transform Encoder and Transform Decoder



Motivation of Transform Coding

Exploitation of Statistical Dependencies

- Typically, the signal energy is concentrated in a few transform coefficients
- Coding of a few non-zero coefficients and many zero-valued coefficients can be very efficient (e.g., using arithmetic coding, run-level coding, ...)

➡ Scalar quantization is more effective in transform domain

Efficient trade-off between Coding Efficiency and Complexity

- Vector Quantization: Searching through codebook for best matching vector

➡ Transform and scalar quantization: Substantial reduction in complexity

Suitable for Quantization using Perceptual Criteria

- Speech & audio coding: Frequency bands might be used to simulate processing of human ear
- Image & video coding: Quantization in transform domain leads to subjective improvement

➡ Removal of perceptually irrelevant signal components

Linear Block Transforms: Analysis Transform

Linear Block Transform

- Each component of the N-dimensional output vector $\boldsymbol{u}$ represents a linear combination of the N components of the N-dimensional input vector $\boldsymbol{s}$

➡ Can be represented as matrix multiplication

Linear Analysis Transform

- Block of samples $\boldsymbol{s}$ is converted into vector of transform coefficients $\boldsymbol{u}$

$$\boldsymbol{u} = \boldsymbol{A} \cdot \boldsymbol{s}$$

- Extended notation

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} & & \\ & \boldsymbol{A} & \\ & & \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{N-1} \end{bmatrix}$$

Linear Block Transforms: Synthesis Transform

Linear Synthesis Transform

- Vector of reconstructed transform coefficients $\boldsymbol{u}'$ is converted into block of reconstructed samples $\boldsymbol{s}'$

$$\boldsymbol{s}' = \boldsymbol{B} \cdot \boldsymbol{u}'$$

- Extended notation

$$\begin{bmatrix} s'_0 \\ s'_1 \\ s'_2 \\ \vdots \\ s'_{N-1} \end{bmatrix} = \begin{bmatrix} & & \\ & \boldsymbol{B} & \\ & & \end{bmatrix} \cdot \begin{bmatrix} u'_0 \\ u'_1 \\ u'_2 \\ \vdots \\ u'_{N-1} \end{bmatrix}$$

➡ Interpretation: Vector of reconstructed samples $\boldsymbol{s}'$ is represented as a linear combination of column vectors $\{\boldsymbol{b}_k\}$ of the synthesis matrix $\boldsymbol{B}$

$$\boldsymbol{s}' = u'_0 \cdot \boldsymbol{b}_0 + u'_1 \cdot \boldsymbol{b}_1 + u'_2 \cdot \boldsymbol{b}_1 + \ldots + u'_{N-1} \cdot \boldsymbol{b}_{N-1}$$

Interpretation of Synthesis Transform

Synthesis Transform

- Reconstructed block of samples $\boldsymbol{s'}$

$$\underbrace{\begin{bmatrix} s'_0 \\ s'_1 \\ s'_2 \\ \vdots \\ s'_{N-1} \end{bmatrix}}_{\boldsymbol{s'}} = u'_0 \cdot \underbrace{\begin{bmatrix} b_{00} \\ b_{01} \\ b_{02} \\ \vdots \\ b_{0,N-1} \end{bmatrix}}_{\boldsymbol{b}_0} + u'_1 \cdot \underbrace{\begin{bmatrix} b_{10} \\ b_{11} \\ b_{12} \\ \vdots \\ b_{1,N-1} \end{bmatrix}}_{\boldsymbol{b}_1} + u'_2 \cdot \underbrace{\begin{bmatrix} b_{20} \\ b_{21} \\ b_{22} \\ \vdots \\ b_{2,N-1} \end{bmatrix}}_{\boldsymbol{b}_2} + \cdots$$

➡ Reconstructed transform coefficients $\{u'_k\}$ represent weighting factors for basis vectors $\{\boldsymbol{b}_k\}$ (i.e., columns) of synthesis transform matrix $\boldsymbol{B}$

Analysis Transform for most relevant case $\boldsymbol{A} = \boldsymbol{B}^{-1}$

➡ Decomposition of sample vector $\boldsymbol{s}$ into basis vectors $\{\boldsymbol{b}_k\}$

➡ Transform coefficients u_k represent the corresponding weighting factors

Example for Possible Basis Vectors (of size 4)

$$\boldsymbol{b}_0 = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b}_1 = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \qquad \boldsymbol{b}_2 = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{b}_3 = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

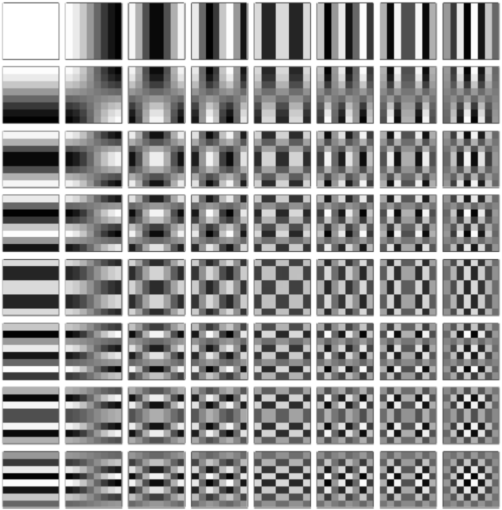
➡ Synthesis matrix $\boldsymbol{B}$

$$\boldsymbol{B} = \begin{bmatrix} | & | & | & | \\ \boldsymbol{b}_0 & \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \\ | & | & | & | \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

➡ Associated analysis matrix $\boldsymbol{A}$ (typical choice)

$$\boldsymbol{A} = \boldsymbol{B}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Example: Typical Basis Functions for 8×8 Image Blocks



Perfect Reconstruction Property

Without Quantization

- Transform coefficients are lossless coded: $\boldsymbol{u}' = \boldsymbol{u}$
- ➡ Optimal synthesis transform: $\boldsymbol{B} = \boldsymbol{A}^{-1}$
- ➡ Reconstructed samples are equal to source samples

$$\boldsymbol{s}' = \boldsymbol{B}\,\boldsymbol{u} = \boldsymbol{B}\,\boldsymbol{A}\,\boldsymbol{s} = \boldsymbol{A}^{-1}\boldsymbol{A}\,\boldsymbol{s} = \boldsymbol{s}$$

Optimal Synthesis Transform (in presence of quantization)

- Optimality: Minimum MSE distortion among all synthesis transforms for given analysis transform $\boldsymbol{A}$
- $\boldsymbol{B} = \boldsymbol{A}^{-1}$ is optimal if
  - $\boldsymbol{A}$ is invertible and produces independent transform coefficients
  - all component quantizers are centroid quantizers
- If above conditions are not fulfilled, a synthesis transform $\boldsymbol{B} \neq \boldsymbol{A}^{-1}$ may reduce the distortion

➡ **In Practice: Use linear transforms with $\boldsymbol{B} = \boldsymbol{A}^{-1}$**

Unitary Transforms

Unitary Matrix

- Inverse matrix is equal to its conjugate transpose

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^{\dagger} = (\boldsymbol{A}^{*})^{\mathrm{T}}$$

➡ Unitary transforms preserve length of vectors: $\|\boldsymbol{A}\cdot\boldsymbol{s}\|_2 = \|\boldsymbol{s}\|_2$

$$\begin{aligned}
\|\boldsymbol{u}\|_2^2 &= \sum_k |u_k|^2 = \sum_k u_k^* \cdot u_k = (\boldsymbol{u}^*)^{\mathrm{T}}\,\boldsymbol{u} \\
&= \boldsymbol{u}^{\dagger}\cdot\boldsymbol{u} = (\boldsymbol{A}\boldsymbol{s})^{\dagger}\cdot(\boldsymbol{A}\boldsymbol{s}) = \boldsymbol{s}^{\dagger}\cdot\boldsymbol{A}^{\dagger}\cdot\boldsymbol{A}\cdot\boldsymbol{s} \\
&= \boldsymbol{s}^{\dagger}\cdot\left(\boldsymbol{A}^{-1}\cdot\boldsymbol{A}\right)\cdot\boldsymbol{s} = \boldsymbol{s}^{\dagger}\cdot\boldsymbol{s} \\
&= \sum_k s_k^*\cdot s_k = \sum_k |s_k|^2 \\
&= \|\boldsymbol{s}\|_2^2
\end{aligned}$$

Orthogonal Transforms

Orthogonal Matrix

- Special case of unitary matrix: All matrix elements are real values

➜ Inverse matrix is equal to the transpose

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}}$$

Basis Vectors

- Columns of synthesis matrix $\boldsymbol{B}$

➜ Rows of analysis matrix $\boldsymbol{A} = \boldsymbol{B}^{\mathrm{T}}$

$$\boldsymbol{A} = \begin{bmatrix} \text{——} & \boldsymbol{b}_0 & \text{——} \\ \text{——} & \boldsymbol{b}_1 & \text{——} \\ \text{——} & \boldsymbol{b}_2 & \text{——} \\ & \vdots & \\ \text{——} & \boldsymbol{b}_{N-1} & \text{——} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} | & | & | & & | \\ \boldsymbol{b}_0 & \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_{N-1} \\ | & | & | & & | \end{bmatrix}$$

Orthonormal Basis

Property of Unitary Transforms

- Consider product of analysis and synthesis matrix: $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{B}^{\dagger}\boldsymbol{B}$

$$\begin{bmatrix} \text{---} & \boldsymbol{b}_0^* & \text{---} \\ \text{---} & \boldsymbol{b}_1^* & \text{---} \\ \text{---} & \boldsymbol{b}_2^* & \text{---} \\ & \vdots & \\ \text{---} & \boldsymbol{b}_{N-1}^* & \text{---} \end{bmatrix} \begin{bmatrix} | & | & | & & | \\ \boldsymbol{b}_0 & \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_{N-1} \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

➡ Basis vectors $\boldsymbol{b}_k$ are orthogonal to each other

➡ Basis vectors $\boldsymbol{b}_k$ have a length equal to 1

➡ **Basis vectors of unitary matrices form an orthonormal basis**

Geometric Interpretation

- Rotation (and possible reflection) of coordinate system

Example of Orthogonal Transform for $N = 2$

- Vector of two samples $\boldsymbol{s} = (s_0, s_1)^{\mathrm{T}}$
- Synthesis transform matrix

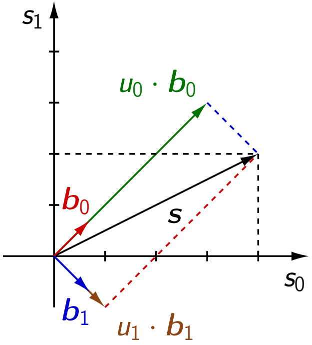
$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_0 & \boldsymbol{b}_1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- Representation of signal vector

$$\boldsymbol{s} = u_0 \cdot \boldsymbol{b}_0 + u_1 \cdot \boldsymbol{b}_1$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = u_0 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + u_1 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



➡ Forward transform: Project signal vector onto basis vectors

$$u_0 = \boldsymbol{b}_0^{\mathrm{T}} \cdot \boldsymbol{s} = 3\sqrt{2} \qquad \text{and} \qquad u_1 = \boldsymbol{b}_1^{\mathrm{T}} \cdot \boldsymbol{s} = \sqrt{2}$$

Unitary Transforms: MSE Distortion

Conservation of MSE distortion

- Remember: Conservation of signal energy / vector length

$$\|\boldsymbol{u}\|_2^2 = \|\boldsymbol{A}\cdot\boldsymbol{s}\|_2^2 = \|\boldsymbol{s}\|_2^2$$

➡ Consequence for MSE distortion

$$\begin{aligned} d_N(\boldsymbol{u},\boldsymbol{u}') &= \frac{1}{N}\|\boldsymbol{u}-\boldsymbol{u}'\|_2^2 \\ &= \frac{1}{N}\left\|\boldsymbol{A}\boldsymbol{s}-\boldsymbol{B}^{-1}\boldsymbol{s}'\right\|_2^2 = \frac{1}{N}\left\|\boldsymbol{A}\left(\boldsymbol{s}-\boldsymbol{s}'\right)\right\|_2^2 \\ &= \frac{1}{N}\|\boldsymbol{s}-\boldsymbol{s}'\|_2^2 = d_N(\boldsymbol{s},\boldsymbol{s}') \end{aligned}$$

Main Reason for using Unitary Transforms

➡ Minimization of MSE distortion $d_N(\boldsymbol{u},\boldsymbol{u}')$ in transform domain also minimizes MSE distortion $d_N(\boldsymbol{s},\boldsymbol{s}')$ in original signal space

➡ **Enables independent scalar quantization of transform coefficients**

Unitary Transforms: Covariance Matrix

Covariance of Transform Coefficients

- Covariance matrix of transform coefficients (general case: complex values)

$$
\begin{aligned}
\boldsymbol{C}_{UU} &= \mathrm{E}\Big\{ (\boldsymbol{U} - \mathrm{E}\{\,\boldsymbol{U}\,\})\,(\boldsymbol{U} - \mathrm{E}\{\,\boldsymbol{U}\,\})^{\dagger} \Big\} \\
&= \mathrm{E}\Big\{ \boldsymbol{A}\,(\boldsymbol{S} - \mathrm{E}\{\,\boldsymbol{S}\,\})\,(\boldsymbol{S} - \mathrm{E}\{\,\boldsymbol{S}\,\})^{\dagger}\,\boldsymbol{A}^{\dagger} \Big\} \\
&= \boldsymbol{A} \cdot \mathrm{E}\Big\{ (\boldsymbol{S} - \mathrm{E}\{\,\boldsymbol{S}\,\})\,(\boldsymbol{S} - \mathrm{E}\{\,\boldsymbol{S}\,\})^{\dagger} \Big\} \cdot \boldsymbol{A}^{\dagger} \\
&= \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\dagger} \\
&= \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{-1}
\end{aligned}
$$

➡ Transform matrix $\boldsymbol{A}$ can be chosen in a way that (linear) statistical dependencies are reduced

➡ **Possible to increase efficiency of scalar quantization**
(if source contains linear statistical dependencies)

Unitary Transforms: Variances

Variances of Transform Coefficients

- Sum of variances: Trace of autocovariance matrix

$$\boldsymbol{C}_{UU} = \begin{bmatrix} \sigma_0^2 & x & x & \cdots & x \\ x & \sigma_1^2 & x & \cdots & x \\ x & x & \sigma_2^2 & \cdots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & \cdots & \sigma_{N-1}^2 \end{bmatrix} = \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{-1}$$

- Trace of a matrix is similarity-invariant

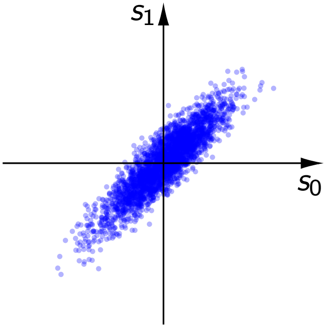
$$\mathrm{tr}(\boldsymbol{X}) = \mathrm{tr}(\boldsymbol{Q}\,\boldsymbol{X}\,\boldsymbol{Q}^{-1})$$

➡ **The arithmetic mean of the transform coefficient variances is equal to source variance**

$$\frac{1}{N}\sum_{k=0}^{N-1} \sigma_k^2 = \sigma_S^2$$

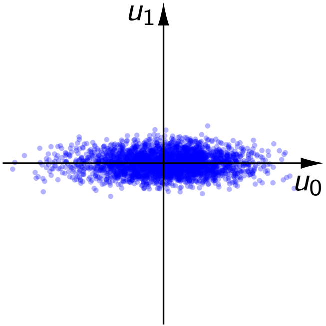
Effect of Orthogonal Transform for Correlated Sources

- 2d signal vectors of Gauss-Markov source with $\varrho = 0.9$



$$\boldsymbol{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

rotation by
$\phi = -45^\circ$

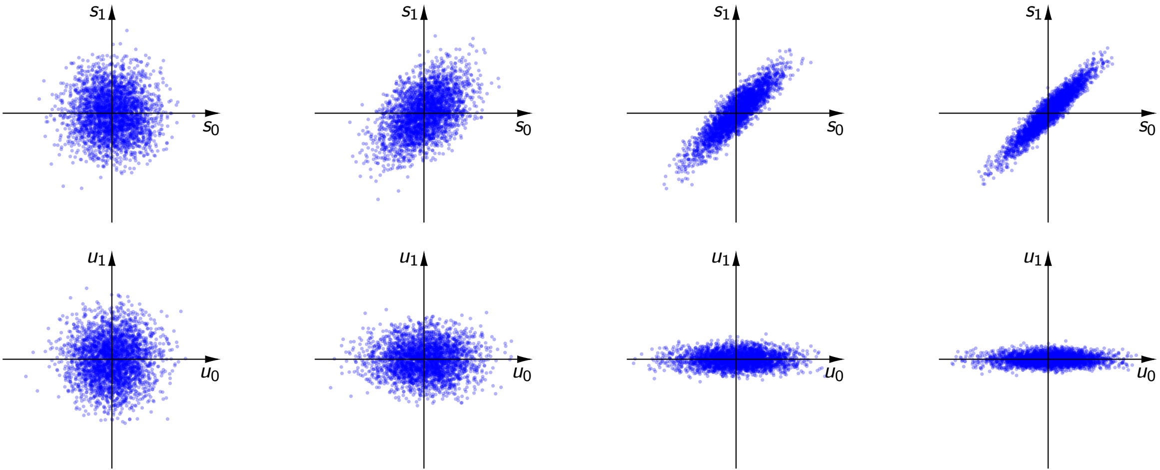


➡ Uneven distribution of transform coefficient variances: $\sigma_0^2 > \sigma_1^2$

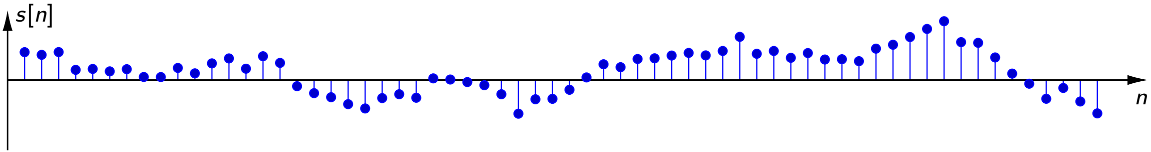
➡ Most signal energy is concentrated in first transform coefficient

Gauss-Markov Examples for $N = 2$

$\varrho = 0.00$ | $\varrho = 0.50$ | $\varrho = 0.90$ | $\varrho = 0.95$

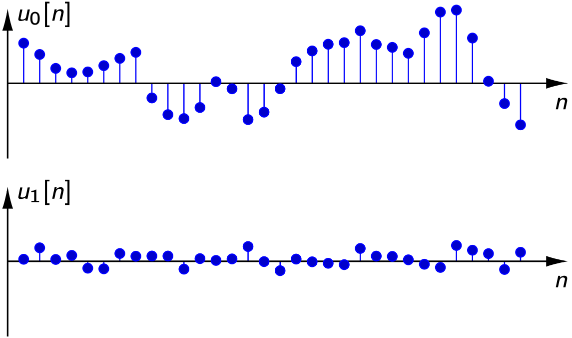


Example for Waveforms: Gauss-Markov with $\varrho = 0.95$



$$\begin{pmatrix} u_0[n] \\ u_1[n] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} s[2n] \\ s[2n+1] \end{pmatrix}$$

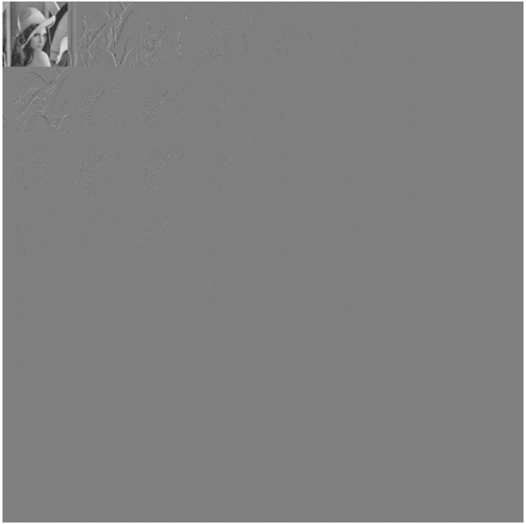
most signal energy is concentrated in u_0



Example for Images: 2×2 Block Transform (sorted Coefficients)

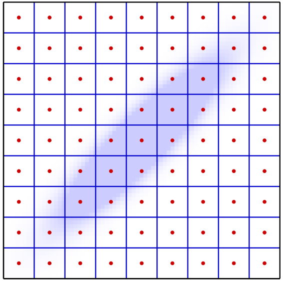
Example for Images: 4×4 Block Transform (sorted Coefficients)

Example for Images: 8×8 Block Transform (sorted Coefficients)



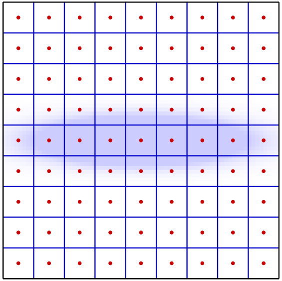
Transform Coding as Constrained Vector Quantizer

scalar quantization



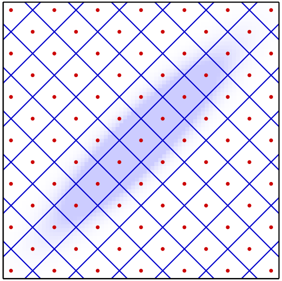
quantization cells

transform coding



quantization cells
in transform domain

transform coding



quantization cells
in signal space

- Quantization cells are:
  - hyper-rectangles as in conventional scalar quantization
  - but rotated and aligned with the transform basis vectors

➡ On average: Value of second quantization index is reduced (for correlated sources)

➡ Indicates improved coding efficiency for correlated sources (exploits memory advantage)

Bit Allocation for Transform Coefficients

- Given: Orthogonal transform with $\boldsymbol{A}$ and $\boldsymbol{B} = \boldsymbol{A}^{\mathrm{T}}$
- Operational distortion-rate function of scalar quantizers (general form)

$$D_k(R_k) = \sigma_k^2 \cdot g_k(R_k)$$

- Overall MSE distortion D and bit rate R (transform size N)

$$D = \frac{1}{N} \sum_{k=0}^{N-1} D_k(R_k) \qquad \text{and} \qquad R = \frac{1}{N} \sum_{k=0}^{N-1} R_k$$

Bit allocation

- Overall rate-distortion performance $D(R)$ depends on bit distribution among transform coefficents $R \mapsto \{R_0, R_1, \cdots\}$

➡ Optimal bit allocation: Solution of optimization problem

$$\min \; D(R_0, R_1, \cdots) \qquad \text{subject to} \qquad \frac{1}{N} \sum_k R_k = R$$

Bit Allocation for Transform Coefficients

- Constrained optimization problem

$$\min_{R_0, R_1, \cdots} \; D(R) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(R_k) \qquad \text{subject to} \qquad \frac{1}{N} \sum_{k=0}^{N-1} R_k = R$$

with $D_k(R_k)$ being the operational distortion-rate functions the scalar component quantizers

➡ Reformulate as unconstrained minimization problem using the technique of Lagrange multipliers (minimize $D + \lambda R$)

$$\min_{R_0, R_1, \cdots} \; \left(\frac{1}{N} \sum_{k=0}^{N-1} D_k(R_k) \right) + \lambda \cdot \left(\frac{1}{N} \sum_{k=0}^{N-1} R_k \right)$$

➡ Set derivatives with respect to R_k equal to 0

$$\frac{\partial}{\partial R_k} \left(D + \lambda R \right) \stackrel{!}{=} 0$$

Optimal Bit Allocation: Pareto Condition

- Minimize Lagrangian cost function $D + \lambda R$

$$\frac{\partial}{\partial R_k}\left(\frac{1}{N}\sum_{i=0}^{N-1} D_i(R_i) + \frac{\lambda}{N}\sum_{i=0}^{N-1} R_i\right) \overset{!}{=} 0$$

$$\frac{1}{N}\cdot\frac{\partial}{\partial R_k} D_k(R_k) + \frac{\lambda}{N} \overset{!}{=} 0$$

➡ Solution: **Pareto condition**

$$\boxed{\frac{\partial D_k(R_k)}{\partial R_k} = -\lambda = \text{const}}$$

➡ All component quantizers have to be operated at the same slope of their operational distortion-rate function

➡ Interpretation: Move bits from coefficients with small distortion reduction per bit to coefficients with larger distortion reduction per bit

High-Rate Approximation: Bit Allocation

High Rates

- All component quantizers are operated at high component rates R_k
- High-rate approximation of distortion-rate function for component quantizers

$$D_k(R_k) = \varepsilon_k^2 \cdot \sigma_k^2 \cdot 2^{-2R_k}$$

where ε_k^2 depends on transform coefficient distribution and quantizer

Optimal Bit Allocation at High Rates

- Pareto condition

$$\frac{\partial}{\partial R_k} D_k(R_k) = -2\,\ln 2\,\varepsilon_k^2\,\sigma_k^2\,2^{-2R_k} = -2\,\ln 2\,D_k(R_k) = -\lambda = \text{const}$$

➡ **All component quantizers are operated at the same distortion**

$$D_k(R_k) = D$$

High-Rate Approximation: Bit Allocation

Optimal Bit Allocation

- All component quantizers are operated at the same distortion

$$D_k(R_k) = \varepsilon_k^2 \cdot \sigma_k^2 \cdot 2^{-2R_k} = D$$

➡ Bit allocation rule

$$R_k(D) = \frac{1}{2} \log_2\left(\frac{\varepsilon_k^2 \, \sigma_k^2}{D}\right)$$

Overall Operational Rate-Distortion Function

- Use result of optimal bit allocation

$$R(D) = \frac{1}{N} \sum_{k=0}^{N-1} R_k(D) = \frac{1}{2N} \sum_{k=0}^{N-1} \log_2\left(\frac{\varepsilon_k^2 \, \sigma_k^2}{D}\right)$$

High-Rate Approximation: Distortion-Rate Function

- Operational Rate-Distortion Function

$$R(D) = \frac{1}{2N} \sum_{k=0}^{N-1} \log_2\left(\frac{\varepsilon_k^2 \, \sigma_k^2}{D} \right) = \frac{1}{2} \log_2\left(\frac{1}{D} \left(\prod_{k=0}^{N-1} \varepsilon_k^2 \right)^{\frac{1}{N}} \left(\prod_{k=0}^{N-1} \sigma_k^2 \right)^{\frac{1}{N}} \right)$$

- Define geometric means

$$\tilde{\sigma}^2 = \left(\prod_{k=0}^{N-1} \sigma_k^2 \right)^{\frac{1}{N}} \qquad \text{and} \qquad \tilde{\varepsilon}^2 = \left(\prod_{k=0}^{N-1} \varepsilon_k^2 \right)^{\frac{1}{N}}$$

➡ **High-rate rate-distortion / distortion-rate function** (for optimal bit allocation)

$$\boxed{R(D) = \frac{1}{2} \log_2\left(\frac{\tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2}{D} \right)} \qquad \text{and} \qquad \boxed{D(R) = \tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2 \cdot 2^{-2R}}$$

High-Rate Approximation for Gaussian Sources

Transform Coding for Gaussian Sources

- Any linear combination of Gaussian random variables is also a Gaussian random variable

➡ All transform coefficients represent Gaussian random variables

Transform Coding for Gaussian Sources using Optimal Scalar Quantizers

- High-rate distortion-rate function of entropy-constrained scalar quantizers

$$D_k(R_k) = \frac{\pi e}{6} \cdot \sigma_k^2 \cdot 2^{-2R_k}$$

➡ Overall high-rate distortion-rate function for Gaussian sources

$$D_G(R) = \frac{\pi e}{6} \cdot \tilde{\sigma}^2 \cdot 2^{-2R}$$

➡ Improvement relative to scalar quantization for uneven distribution of transform coefficient variances

Transform Coding Gain at High Rates

Transform Coding Gain

- Ratio of distortion for scalar quantization and transform coding
- Transform coding gain at high rates

$$G_T = \frac{D_{SQ}(R)}{D_{TQ}(R)} = \frac{\varepsilon_S^2 \cdot \sigma_S^2 \cdot 2^{-2R}}{\tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2 \cdot 2^{-2R}} = \frac{\varepsilon_S^2 \cdot \sigma_S^2}{\tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2}$$

Transform Coding Gain for Gaussian Sources

- High-rate transform coding gain for Gaussian sources

$$G_T = \frac{\sigma_S^2}{\tilde{\sigma}^2} = \frac{\frac{1}{N}\sum_{k=0}^{N-1} \sigma_k^2}{\sqrt[N]{\prod_{k=0}^{N-1} \sigma_k^2}}$$

➡ Ratio of arithmetic and geometric mean of the transform coefficient variances

➡ Transform coding gain is maximized if the geometric mean $\tilde{\sigma}^2$ of variances is minimized

Example: Transform Coding with $N = 2$ for Zero-Mean Gaussian

- Input vector and transform matrix

$$\boldsymbol{s} = \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

→ Transformation

$$\boldsymbol{u} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \boldsymbol{A} \cdot \boldsymbol{s} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \end{bmatrix}$$

→ Transform coefficients

$$u_0 = \frac{1}{\sqrt{2}}(s_0 + s_1) \qquad \text{and} \qquad u_0 = \frac{1}{\sqrt{2}}(s_0 - s_1)$$

- Inverse transformation

$$\boldsymbol{B} = \boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Example: Transform Coding with $N = 2$ for Zero-Mean Gaussian

- Transform coefficients

$$u_0 = \frac{1}{\sqrt{2}}(s_0 + s_1) \qquad \text{and} \qquad u_0 = \frac{1}{\sqrt{2}}(s_0 - s_1)$$

- Variance of transform coefficients

$$\begin{aligned} \sigma_0^2 &= \mathrm{E}\{\, U_0^2 \,\} = \frac{1}{2}\mathrm{E}\{\, (S_0 + S_1)^2 \,\} = \frac{1}{2}\Big(\mathrm{E}\{\, S_0^2 \,\} + \mathrm{E}\{\, S_1^2 \,\} + 2\mathrm{E}\{\, S_0 S_1 \,\}\Big) \\ &= \frac{1}{2}\left(\sigma_S^2 + \sigma_S^2 + 2\sigma_S^2 \varrho\right) = \sigma_S^2\,(1 + \varrho) \end{aligned}$$

$$\sigma_1^2 = \mathrm{E}\{\, U_1^2 \,\} = \sigma_S^2\,(1 - \varrho)$$

- Cross-correlation of transform coefficients

$$E\{U_0 U_1\} = \frac{1}{2}\mathrm{E}\{\, (S_0 + S_1)\,(S_0 - S_1)\,\} = \frac{1}{2}\mathrm{E}\{\, S_0^2 - S_1^2 \,\} = \sigma_S^2 - \sigma_S^2 = 0$$

Example: Transform Coding with $N = 2$ for Zero-Mean Gaussian

- High rate distortion-rate functions of component quantizers

$$
\begin{aligned}
D_0(R_0) &= \varepsilon^2\,\sigma_0^2\,2^{-2R_0} = \varepsilon^2\,\sigma_S^2\,(1+\varrho)\,2^{-2R_0} \\
D_1(R_1) &= \varepsilon^2\,\sigma_1^2\,2^{-2R_1} = \varepsilon^2\,\sigma_S^2\,(1-\varrho)\,2^{-2R_1}
\end{aligned}
$$

- Optimal bit allocation: Pareto condition at high rates $D_0(R_0) = D_1(R_1)$

$$
\begin{aligned}
\varepsilon^2\,\sigma_S^2\,(1+\varrho)\,2^{-2R_0} &= \varepsilon^2\,\sigma_S^2\,(1-\varrho)\,2^{-2R_1} \\
\log_2(1+\varrho) - 2R_0 &= \log_2(1-\varrho) - 2R_1
\end{aligned}
$$

➡ Using $R = \frac{1}{2}(R_0 + R_1)$ ➡ $R_1 = 2R - R_0$

$$
\begin{aligned}
\log_2(1+\varrho) - 2R_0 &= \log_2(1-\varrho) - 4R + 2R_0 \\
4R_0 &= 4R + \log_2(1+\varrho) - \log_2(1-\varrho) \\
R_0 &= R + \frac{1}{4}\log_2\left(\frac{1+\varrho}{1-\varrho}\right)
\end{aligned}
$$

Example: Transform Coding with $N = 2$ for Zero-Mean Gaussian

- Optimal bit allocation

$$R_0 = R + \frac{1}{4}\log_2\left(\frac{1+\varrho}{1-\varrho}\right) \qquad \text{and} \qquad R_1 = R - \frac{1}{4}\log_2\left(\frac{1+\varrho}{1-\varrho}\right)$$

- Resulting component distortions

$$\begin{aligned} D_0(R) &= \varepsilon^2\,\sigma_s^2\,(1+\varrho)\,2^{-2R-\frac{1}{2}\log_2\left(\frac{1+\varrho}{1-\varrho}\right)} \\ &= \varepsilon^2\,\sigma_s^2\,(1+\varrho)\,2^{-2R}\sqrt{\frac{1-\varrho}{1+\varrho}} = \varepsilon^2\,\sigma_s^2\,\sqrt{1-\varrho^2}\,2^{-2R} \end{aligned}$$

$$\begin{aligned} D_1(R) &= \varepsilon^2\,\sigma_s^2\,(1-\varrho)\,2^{-2R+\frac{1}{2}\log_2\left(\frac{1+\varrho}{1-\varrho}\right)} \\ &= \varepsilon^2\,\sigma_s^2\,(1-\varrho)\,2^{-2R}\sqrt{\frac{1+\varrho}{1-\varrho}} = \varepsilon^2\,\sigma_s^2\,\sqrt{1-\varrho^2}\,2^{-2R} \end{aligned}$$

Example: Transform Coding with $N = 2$ for Zero-Mean Gaussian

- Component distortions

$$D_0(R) = D_1(R) = \varepsilon^2\,\sigma_s^2\,\sqrt{1-\varrho^2}\,2^{-2R}$$

- Distortion rate function

$$D(R) = \frac{1}{2}\big(D_0(R) + D_1(R)\big) = \varepsilon^2\,\sigma_s^2\,\sqrt{1-\varrho^2}\,2^{-2R}$$

- Geometric mean of variances

$$\tilde{\sigma}^2 \;=\; \sqrt{\sigma_0^2 \cdot \sigma_1^2} = \sigma_S^2 \cdot \sqrt{(1+\varrho)(1-\varrho)} = \sigma_S^2 \cdot \sqrt{1-\varrho^2}$$

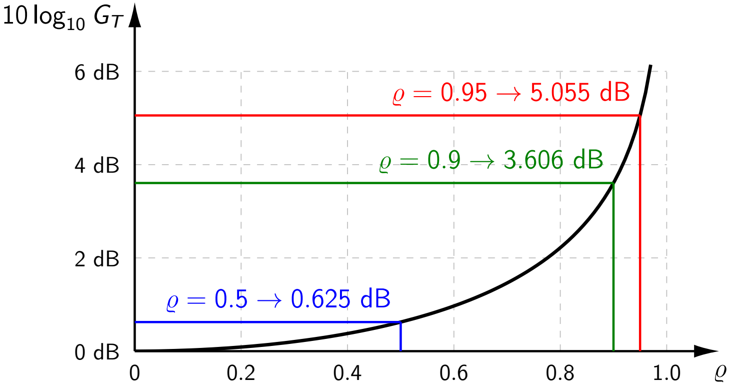
➡ Yields same expression for distortion rate function

$$D(R) = \varepsilon^2\,\tilde{\sigma}^2\,2^{-2R} = \varepsilon^2\,\sigma_S^2\,\sqrt{1-\varrho^2}\,2^{-2R}$$

Example: Transform Coding with $N = 2$ for Zero-Mean Gaussian

- Transform coding gain for $N = 2$

$$G_T = \frac{\varepsilon^2\, \sigma_S^2\, 2^{-2R}}{\varepsilon^2\, \sigma_S^2\, \sqrt{1-\varrho^2}\, 2^{-2R}} = \frac{1}{\sqrt{1-\varrho^2}}$$



Summary of Lecture

Transform Coding

- Linear unitary/orthogonal transform of block/vector of N consecutive samples
- Scalar quantization of resulting transform coefficients
- Inverse linear transform of reconstructed transform coefficients

Orthogonal Block Transforms

- Inverse transform matrix = Transpose of forward transform matrix

➡ Coordinate axes remain orthogonal to each other (independent quantization)

➡ MSE distortion: Same in transform domain and signal space

Bit Allocation

- Optimal bit allocation: Pareto condition (same slope for all $D_k(R_K)$)
- For high rates: Optimum bit allocation yields equal component distortions $D_k = D$

Summary of Lecture

High-Rate Approximations

- Distortion-rate function of transform coding

$$D(R) = \tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2 \cdot 2^{-2R}$$

- Transform coding gain for Gaussian sources

$$G_T = \frac{\bar{\sigma}^2}{\tilde{\sigma}^2} = \frac{\text{arithmetic mean of variances}}{\text{geometric mean of variances}}$$

➞ Goal of transform: Compaction of signal energy in few transform coefficients

Open Questions

- What is the optimal transform for a given sources ?
- Practical aspects of transform coding

Exercise 1: Orthogonal Transforms of Size $N = 2$ (part I)

If we neglect possible reflections of coordinate axes, all orthogonal transforms for 2-d vectors can be specified by

$$\boldsymbol{A} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

where α is an arbitrary rotation angle.

Consider a zero-mean Gaussian process with variance σ_S^2 and the first-order correlation coefficient ϱ.

(a) Calculate the variances σ_0^2 and σ_1^2 of the resulting transform coefficients as function of ϱ and α.

(b) Calculate the covariance σ_{01}^2 between the resulting transform coefficients as function of ϱ and α.

(c) Consider an even rate distribution $R_0 = R_1 = R$ and determine the associated high-rate distortion-rate function. Does transform coding improve the coding efficiency relative to scalar quantization for this case?

Exercise 1: Orthogonal Transforms of Size $N = 2$ (part II)

(d) Given is the overall rate $R = (R_0 + R_1)/2$. Determine the rate distribution (R_0, R_1) for which the overall distortion $D = (D_0 + D_1)/2$ is minimized (assume that the high rate approximation for scalar quantization of the transform coefficients is valid).

(e) Determine the overall distortion-rate function for optimal rate allocation (and high rates).

(f) Determine the high-rate transform coding gain, which is given by

$$G_T = \frac{D_{\text{scalar quantization}}(R)}{D_{\text{transform coding}}(R)}$$

(g) For what rotation angles is the high-rate transform coding gain maximized (or the distortion minimized)?

Does the optimal rotation angle depend on the correlation coefficient ϱ?

Exercise 2: Implement a PSNR Tool for PPM Images

Implement a tool for measuring PSNRs between two PPM images

- Input to the tool shall be two images in PPM format (original and reconstructed)
- The tool should output the following four Peak-Signal-to-Noise Ratios (PSNR measures)
  - ➡ PSNR of red component, PSNR of green component, PSNR of blue component
  - ➡ Average of the red, green, and blue PSNR

Test the tool by

- Coding one of our test images with JPEG (e.g., using "convert test.ppm test.jpg")
- Reconstructing the JPEG-coded image into the ppm format (e.g., using "convert test.jpg rec.ppm")
- Measuring the PSNRs between the original and reconstructed image using the implemented tool

The PSNR for a color component $c[x, y]$ and its reconstruction $c'[x, y]$ is defined as follows

$$\text{PSNR} = 10 \cdot \log_{10}\left(\frac{255^2}{\text{MSE}}\right) \qquad \text{with} \qquad \text{MSE} = \frac{1}{\text{width} \cdot \text{height}} \sum_{x,y} \left(c'[x,y] - c[x,y]\right)^2$$