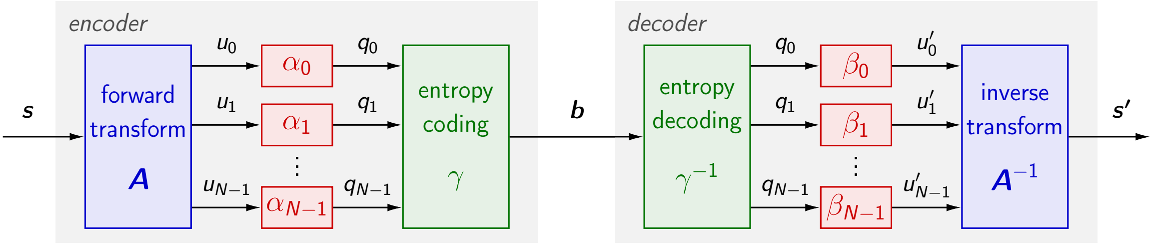
The Karhunen Loève Transform

$$C_{SS} = \begin{bmatrix} 1.000 & 0.900 & 0.810 & 0.729 \\ 0.900 & 1.000 & 0.900 & 0.810 \\ 0.810 & 0.900 & 1.000 & 0.900 \\ 0.729 & 0.810 & 0.900 & 1.000 \end{bmatrix} \xrightarrow{\boldsymbol{u} = \boldsymbol{A}_{KLT} \cdot \boldsymbol{s}} C_{UU} = \begin{bmatrix} 3.527 & 0 & 0 & 0 \\ 0 & 0.310 & 0 & 0 \\ 0 & 0 & 0.102 & 0 \\ 0 & 0 & 0 & 0.061 \end{bmatrix}$$

Last Lecture: Basic Concept Transform Coding

- Transform removes (or reduces) linear dependencies between samples before scalar quantization
- For correlated sources: Scalar quantization in transform domain is more efficient



Encoder (block-wise)

➡ Forward transform: $\boldsymbol{u} = \boldsymbol{A} \cdot \boldsymbol{s}$

➡ Scalar quantization: $q_k = \alpha_k(u_k)$

➡ Entropy coding: $\boldsymbol{b} = \gamma(\{q_k\})$

Decoder (block-wise)

➡ Entropy decoding: $\{q_k\} = \gamma^{-1}(\boldsymbol{b})$

➡ Inverse quantization: $u'_k = \beta_k(q_k)$

➡ Inverse transform: $\boldsymbol{s}' = \boldsymbol{A}^{-1} \cdot \boldsymbol{u}'$

Last Lecture: Orthogonal Block Transforms

- Transform matrix has property: $\boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}}$ (special case of unitary matrix)

$$\boldsymbol{A} = \begin{bmatrix} \text{---} & \boldsymbol{b}_0 & \text{---} \\ \text{---} & \boldsymbol{b}_1 & \text{---} \\ \text{---} & \boldsymbol{b}_2 & \text{---} \\ & \vdots & \\ \text{---} & \boldsymbol{b}_{N-1} & \text{---} \end{bmatrix} \qquad \boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} | & | & | & & | \\ \boldsymbol{b}_0 & \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_{N-1} \\ | & | & | & & | \end{bmatrix}$$

➡ Basis vectors (rows of $\boldsymbol{A}$, columns of $\boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}}$) form an orthonormal basis

➡ Geometric interpretation: Rotation (and potential reflection) in N-dimensional signal space

Properties of Orthogonal Transforms

- Preservation of signal energy / vector length: $||\boldsymbol{A} \cdot \boldsymbol{s}||_2 = ||\boldsymbol{s}||_2$
- Same MSE distortion in sample and transform space: $||\boldsymbol{u}' - \boldsymbol{u}||_2^2 = ||\boldsymbol{s}' - \boldsymbol{s}||_2^2$
- Auto-covariance matrix of transform coefficients: $\boldsymbol{C}_{UU} = \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}}$
- Sum of variances of transform coefficients: $\sum_k \sigma_k^2 = N \cdot \sigma_S^2$

Last Lecture: Bit Allocation and High-Rate Approximations

Bit Allocation of Transform Coefficients

- Optimal bit allocation: Pareto condition

$$\frac{\partial}{\partial R_k} D_k(R_k) = -\lambda = \text{const}$$

High-Rate Approximation

- Optimal bit allocation for high-rate case

$$D_k(R_k) = D = \text{const}$$

- High-rate distortion rate function for transform coding

$$D(R) = \tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2 \cdot 2^{-2R} \qquad \text{with} \qquad \tilde{\varepsilon}^2 = \left(\prod_k \varepsilon_k^2\right)^{\frac{1}{N}}, \quad \tilde{\sigma}^2 = \left(\prod_k \sigma_k^2\right)^{\frac{1}{N}}$$

- High-rate transform coding gain

$$G_T = \frac{D_{SQ}(R)}{D_{TC}(R)} = \frac{\varepsilon_S^2 \cdot \sigma_S^2}{\tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2}, \qquad \text{Gaussian sources:} \;\; G_T = \frac{\sigma_S^2}{\tilde{\sigma}^2} = \frac{\frac{1}{N}\sum_k \sigma_k^2}{\tilde{\sigma}^2}$$

How To Choose The Transform ?

Open Questions

- What is the best orthogonal transform for a given source ?
- Is there a low-complex transform that is close to optimal for typical sources ?

Goal: Minimize overall distortion for a given rate (or vice versa)

- High-rate approximation of distortion-rate function (MSE & optimal bit allocation)

$$D(R) = \tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2 \cdot 2^{-2R}$$

➡ High rates: Transform should be designed to minimize geometric mean $\tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2$

Optimal Orthogonal Transform for General Stationary Signals

- Difficult interdependencies between transform and scalar quantization (due to factors ε_k^2)

➡ Optimal transform very difficult to determine (does also depend on bit rate)

➡ Possible: Iterative algorithms for designing both transform and scalar quantizers together

Decorrelating Transforms

Nearly Optimal Transform

- Most important aspect of transform coding: Utilize dependencies between samples
- Linear transform: Can only remove linear dependendencies (correlation)

➞ **Design criterion**: Uncorrelated transform coefficients u_k

$$\forall i, k \neq i : \quad \text{cov}(U_i, U_k) = 0 \qquad \longleftrightarrow \qquad \boldsymbol{C}_{UU} = \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix}$$

Question

Is it possible to find an orthogonal transform matrix $\boldsymbol{A}$ that generates completely decorrelated transform coefficients

$$\boldsymbol{u} = \boldsymbol{A} \cdot \boldsymbol{s}$$

Measure for Energy Compaction

High-Rate Approximations

- High-rate distortion rate function $D(R)$ and transform coding gain G_T

$$D(R) = \tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2 \cdot 2^{-2R} \qquad \Rightarrow \qquad G_T = \frac{\varepsilon_S^2 \cdot \sigma_S^2}{\tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2}$$

- Neglect impact of pdf shape: Assume $\varepsilon_k^2 = \varepsilon_S^2$ (valid for Gaussian sources)
- Remember: Trace of a matrix is similarity-invariant

$$\mathrm{tr}(\boldsymbol{C}_{SS}) = \mathrm{tr}(\boldsymbol{A}\,\boldsymbol{C}_{SS}\,\boldsymbol{A}^{-1}) = \mathrm{tr}(\boldsymbol{C}_{UU})$$

Measure for Energy Compaction of Orthogonal Transforms

➡ Ratio of arithmetic and geometric means for transform coefficient variances

$$G_{EC} = \frac{\sigma_S^2}{\tilde{\sigma}^2} = \frac{\bar{\sigma}^2}{\tilde{\sigma}^2} = \frac{\frac{1}{N}\sum_{k=0}^{N-1}\sigma_k^2}{\sqrt[N]{\prod_{k=0}^{N-1}\sigma_k^2}}$$

➡ Goal of Transform: Maximize energy compaction G_{EC}

The Karhunen Loève Transform (KLT)

Remember: Relationship between auto-covariance matrices for linear transforms ($\boldsymbol{u} = \boldsymbol{A} \cdot \boldsymbol{s}$)

$$\begin{aligned}
\boldsymbol{C}_{UU} &= \mathrm{E}\Big\{ (\boldsymbol{U} - \mathrm{E}\{\, \boldsymbol{U} \,\}) \, (\boldsymbol{U} - \mathrm{E}\{\, \boldsymbol{U} \,\})^{\mathrm{T}} \Big\} \\
&= \mathrm{E}\Big\{ (\boldsymbol{A}\,\boldsymbol{S} - \mathrm{E}\{\, \boldsymbol{A}\,\boldsymbol{S} \,\}) \, (\boldsymbol{A}\,\boldsymbol{S} - \mathrm{E}\{\, \boldsymbol{A}\,\boldsymbol{S} \,\})^{\mathrm{T}} \Big\} \\
&= \boldsymbol{A} \cdot \mathrm{E}\Big\{ (\boldsymbol{S} - \mathrm{E}\{\, \boldsymbol{S} \,\}) \, (\boldsymbol{S} - \mathrm{E}\{\, \boldsymbol{S} \,\})^{\mathrm{T}} \Big\} \cdot \boldsymbol{A}^{\mathrm{T}} \\
&= \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}}
\end{aligned}$$

Karhunen Loève Transform (KLT)

- Orthogonal transform ($\boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}}$) that produces completely decorrelated transform coefficients

➡ Transform matrix $\boldsymbol{A}$ in chosen in a way that auto-covariance matrix

$$\boldsymbol{C}_{UU} = \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}} \qquad \text{becomes a diagonal matrix}$$

➡ Such an orthogonal transform also maximizes the energy compaction $G_{EC} = \bar{\sigma}^2 / \tilde{\sigma}^2$

Basis Vectors of the Karhunen Loève Transform

➡ Required property for the orthogonal transform matrix $\boldsymbol{A}$

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}} &= \boldsymbol{C}_{UU} && \text{(with } \boldsymbol{C}_{UU} \text{ being a diagonal matrix)} \\ (\boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{A}) \cdot \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}} &= \boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{C}_{UU} && \text{(orthogonal transform: } \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}^{-1}\text{)} \\ \boldsymbol{C}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}} &= \boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{C}_{UU} && \text{(rows of } \boldsymbol{A}\text{: basis vectors } \boldsymbol{b}_k\text{)} \end{aligned}$$

$$\boldsymbol{C}_{SS} \cdot \begin{bmatrix} | & | & & | \\ \boldsymbol{b}_0 & \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{N-1} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \boldsymbol{b}_0 & \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{N-1} \\ | & | & & | \end{bmatrix} \cdot \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix}$$

➡ Consider individual columns of matrix equation

$$\boxed{\forall k : \quad \boldsymbol{C}_{SS} \cdot \boldsymbol{b}_k = \sigma_k^2 \cdot \boldsymbol{b}_k}$$

KLT: Determination of Transform Matrix as Eigenvector Problem

Necessary Condition for KLT Basis Vectors

- For each basis vector $\boldsymbol{b}_k$, we have an equation of the form

$$\boldsymbol{C}_{SS} \cdot \boldsymbol{b}_k \;=\; \sigma_k^2 \cdot \boldsymbol{b}_k$$

➡ General form of an **Eigenvector equation**

$$\boldsymbol{C}_{SS} \cdot \boldsymbol{v} \;=\; \xi \cdot \boldsymbol{v} \qquad \text{with} \qquad \begin{array}{ll} \text{eigenvalue} & \xi = \sigma_k^2 \quad \text{and} \\ \text{eigenvector} & \boldsymbol{v} = \boldsymbol{b}_k \end{array}$$

- Note: Eigenvectors $\boldsymbol{v}$ are not unique (can be scaled by any non-zero factor), but basis vectors $\boldsymbol{b}_k$ must have an ℓ_2-norm equal to $\|\boldsymbol{b}_k\|_2 = 1$

KLT Basis Vectors

➡ Basis vectors $\boldsymbol{b}_k = \boldsymbol{v}_k / \|\boldsymbol{v}_k\|_2$ are the **unit-norm eigenvectors** of $\boldsymbol{C}_{SS}$

➡ Transform coefficient variances $\sigma_k^2 = \xi_k$ are given by the associated eigenvalues of $\boldsymbol{C}_{SS}$

Existence and Uniqueness of the Karhunen Loève Transform

Existence of the KLT

- Linear algebra: Symmetric matrices (such as $\boldsymbol{C}_{SS}$) are always orthogonally diagonalizable

➡ KLT exists for all random sources

Uniqueness of the KLT

- Matrix rows (basis vectors) can be permuted or multiplied by -1
- Additional degrees of freedom if two or more eigenvalues are the same

➡ There are multiple KLT transform matrices (with same decorrelation property)

Related Problems

- KLT is also known as **eigenvector transform** or **Hotelling transform**
- KLT is closely related to **principal component analysis** (PCA)
- KLT is a special case of **singular value decomposition** (SVD)

KLT Transform Matrix for Data Compression

- Determination of eigenvectors $\boldsymbol{v}_k$ for given auto-covariance matrix $\boldsymbol{C}_{SS}$
- Unit-norm eigenvectors $\boldsymbol{b}_k$ are sorted in decreasing order of the associated eigenvalues ξ_k

$$\boldsymbol{A} = \begin{bmatrix} — & \boldsymbol{b}_0 & — \\ — & \boldsymbol{b}_1 & — \\ & \vdots & \\ — & \boldsymbol{b}_{N-1} & — \end{bmatrix} \qquad \text{with} \qquad \boldsymbol{b}_k = \frac{\boldsymbol{v}_k}{\|\boldsymbol{v}_k\|_2} \qquad \text{and} \qquad \xi_k \geq \xi_{k+1}$$

➡ Resulting transform coefficients u_k are sorted in decreasing order of their variances $\sigma_k^2 = \xi_k$

$$\boldsymbol{C}_{UU} = \begin{bmatrix} \sigma_0^2 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N-1}^2 \end{bmatrix} \qquad \text{with} \qquad \sigma_k^2 = \xi_k \qquad \text{and} \qquad \xi_k \geq \xi_{k+1}$$

➡ Typical: Order is suitable for entropy coding of quantization indexes (e.g., run-level coding)

KLT Basis Vectors: Determination of Eigenvectors

Calculation of Eigenvalues and Eigenvectors

- General form of Eigenvalue problem

$$
\begin{aligned}
\boldsymbol{C}_{SS} \cdot \boldsymbol{v} &= \xi \cdot \boldsymbol{v} \\
\boldsymbol{C}_{SS} \cdot \boldsymbol{v} &= \xi \cdot \boldsymbol{I} \cdot \boldsymbol{v} \qquad \text{(identity matrix } \boldsymbol{I}\text{)} \\
\left(\boldsymbol{C}_{SS} - \xi \boldsymbol{I} \right) \cdot \boldsymbol{v} &= \boldsymbol{0}
\end{aligned}
$$

Exact Symbolic Calculation

- Homogeneous linear equation system is solvable if and only if $\det(\boldsymbol{C}_{SS} - \xi \boldsymbol{I}) = 0$

➡ Determine the N eigenvalues ξ_k by solving the characteristic polynomial (of degree N)

$$\det\left(\boldsymbol{C}_{SS} - \xi \boldsymbol{I} \right) = 0$$

➡ For each ξ_k, solve linear equation system (any non-trivial solution $\boldsymbol{v}_k$)

$$\left(\boldsymbol{C}_{SS} - \xi_k \boldsymbol{I} \right) \cdot \boldsymbol{v}_k = \boldsymbol{0}$$

KLT Example: Transform Matrix for $N = 2$

- Given: Autocovariance matrix (of order 2) for source samples

$$\boldsymbol{C}_{SS} = \begin{bmatrix} \sigma_S^2 & \varrho\,\sigma_S^2 \\ \varrho\,\sigma_S^2 & \sigma_S^2 \end{bmatrix}$$

➡ Eigenvector equation

$$(\boldsymbol{C}_{SS} - \xi\,\boldsymbol{I})\;\boldsymbol{v} = \begin{bmatrix} \sigma_S^2 - \xi & \varrho\,\sigma_S^2 \\ \varrho\,\sigma_S^2 & \sigma_S^2 - \xi \end{bmatrix} \boldsymbol{v} = \boldsymbol{0}$$

➡ Characteristic polynomial

$$\begin{aligned} \det\left(\boldsymbol{C}_{SS} - \xi\,\boldsymbol{I}\right) &= \left(\sigma_S^2 - \xi\right)^2 - (\varrho\,\sigma_S^2)^2 \\ &= \xi^2 - 2\xi \cdot \sigma_S^2 + \sigma_S^4\,(1 - \varrho^2) = 0 \end{aligned}$$

➡ Eigenvalues

$$\begin{aligned} \xi_{0/1} &= \sigma_S^2 \pm \sqrt{\sigma_S^4 - \sigma_S^4\,(1 - \varrho^2)} = \sigma_S^2 \pm \sqrt{\sigma_S^4\,\varrho^2} \\ \xi_{0/1} &= \sigma_S^2\,(1 \pm \varrho) \end{aligned}$$

KLT Example: Transform Matrix for $N = 2$

➡ Eigenvector equation for first eigenvalue $\xi_0 = \sigma_S^2\,(1+\varrho)$

$$(\boldsymbol{C}_{SS} - \xi_0\,\boldsymbol{I})\;\boldsymbol{v}_0 \;=\; \begin{bmatrix} \sigma_S^2 - \sigma_S^2(1+\varrho) & \varrho\,\sigma_S^2 \\ \varrho\,\sigma_S^2 & \sigma_S^2 - \sigma_S^2(1+\varrho) \end{bmatrix} \boldsymbol{v}_0$$

$$= \sigma_S^2 \begin{bmatrix} -\varrho & \varrho \\ \varrho & -\varrho \end{bmatrix} \boldsymbol{v}_0 = \boldsymbol{0}$$

➡ Equation for vector components $\boldsymbol{v}_0 = (u_0, v_0)$

$$-\varrho \cdot u_0 + \varrho \cdot v_0 = 0 \qquad \Longrightarrow \qquad v_0 = u_0$$

➡ Eigenvector

$$\boldsymbol{v}_0 = \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{with} \qquad \mu \neq 0$$

➡ Basis vector $\boldsymbol{b}_0$ is given by a unit-norm eigenvector

$$\boldsymbol{b}_0 = \frac{\boldsymbol{v}_0}{\|\boldsymbol{v}_0\|_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

KLT Example: Transform Matrix for $N = 2$

- Eigenvector equation for second eigenvalue $\xi_1 = \sigma_S^2\,(1-\varrho)$

$$(\boldsymbol{C}_{SS} - \xi_1\,\boldsymbol{I})\;\boldsymbol{v}_0 \;=\; \begin{bmatrix} \sigma_S^2 - \sigma_S^2(1-\varrho) & \varrho\,\sigma_S^2 \\ \varrho\,\sigma_S^2 & \sigma_S^2 - \sigma_S^2(1-\varrho) \end{bmatrix} \boldsymbol{v}_1$$

$$= \sigma_S^2 \begin{bmatrix} \varrho & \varrho \\ \varrho & \varrho \end{bmatrix} \boldsymbol{v}_1 = \boldsymbol{0}$$

- Equation for vector components $\boldsymbol{v}_1 = (u_1, v_1)$

$$\varrho \cdot u_1 + \varrho \cdot v_1 = 0 \qquad \Longrightarrow \qquad v_1 = -u_1$$

- Eigenvector

$$\boldsymbol{v}_1 = \mu \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{with} \qquad \mu \neq 0$$

- Basis vector $\boldsymbol{b}_1$ is given by a unit-norm eigenvector

$$\boldsymbol{b}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

KLT Example: Transform Matrix for $N = 2$

- Eigenvalues and eigenvectors (with $\mu \neq 0$)

$$\xi_0 = \sigma_S^2\,(1+\varrho) \qquad \xi_1 = \sigma_S^2\,(1-\varrho)$$

$$\boldsymbol{v}_0 = \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_1 = \mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

➡ Basis vectors and transform matrix

$$\boldsymbol{A} = \begin{bmatrix} \text{—}\, \boldsymbol{b}_0 \,\text{—} \\ \text{—}\, \boldsymbol{b}_1 \,\text{—} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

➡ Covariance matrix of transform coefficients

$$\boldsymbol{C}_{SS} = \sigma_S^2 \begin{bmatrix} 1 & \varrho \\ \varrho & 1 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{C}_{UU} = \sigma_S^2 \begin{bmatrix} 1+\varrho & 0 \\ 0 & 1-\varrho \end{bmatrix}$$

➡ Energy compaction

$$G_{EC} = \frac{\bar{\sigma}^2}{\tilde{\sigma}^2} = \frac{1}{\sqrt{1-\varrho^2}}$$

Numerical Algorithms for Eigenvector Computation

Classical Jacobi Algorithm (Carl Gustav Jacob Jacobi, 1846)

- Diagonalize symmetric matrix $\boldsymbol{C}$ by iterative multiplication with elementary rotation matrices $\boldsymbol{R}$

$$\boldsymbol{C}^{(k+1)} = \boldsymbol{R}_k\, \boldsymbol{C}^{(k)}\, \boldsymbol{R}_k^{\mathrm{T}} = \underbrace{\boldsymbol{R}_k\, \boldsymbol{R}_{k-1} \cdots \boldsymbol{R}_0}_{\boldsymbol{A}_k}\, \boldsymbol{C}\, \underbrace{\boldsymbol{R}_0^{\mathrm{T}} \cdots \boldsymbol{R}_{k-1}^{\mathrm{T}} \cdots \boldsymbol{R}_k^{\mathrm{T}}}_{\boldsymbol{A}_k^{\mathrm{T}}}$$

➡ Conceptually simple, but slow convergence (unsuitable for large matrices)

Numerous Advanced Numerical Algorithms (particularly for real symmetric matrices)

- Typical: Two Steps
  1. Transform matrix into Hessenberg / tridiagonal form
  2. Determine eigenvectors of simpler matrix using fast algorithms
- Some examples:
  - Given rotations + divide and conquer
  - Householder transformation + QR algorithm
  - Householder transformation + MRRR algorithm

Auto-Covariance Matrix for AR(1) Sources

AR(1) Sources

- Remember: Auto-covariance function for AR(1) sources

$$\mathrm{cov}(S_k, S_\ell) = \phi_{|k-\ell|} = \mathrm{E}\{\,(S_k-\mu)(S_\ell-\mu)\,\} = \sigma_S^2 \cdot \varrho^{|k-\ell|}$$

with ϱ being the first-order correlation coefficient

➡ N-th order auto-covariance matrix $\boldsymbol{C}_{SS} = \mathrm{E}\{\,(\boldsymbol{S}-\boldsymbol{\mu})\,(\boldsymbol{S}-\boldsymbol{\mu})^{\mathrm{T}}\,\}$

$$\boldsymbol{C}_{SS} = \begin{bmatrix} \phi_0 & \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{N-1} \\ \phi_1 & \phi_0 & \phi_1 & \phi_2 & \cdots & \phi_{N-2} \\ \phi_2 & \phi_1 & \phi_0 & \phi_1 & \cdots & \phi_{N-3} \\ \phi_3 & \phi_2 & \phi_1 & \phi_0 & \cdots & \phi_{N-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{N-1} & \phi_{N-2} & \phi_{N-3} & \phi_{N-4} & \cdots & \phi_0 \end{bmatrix} = \sigma_S^2 \begin{bmatrix} 1 & \varrho & \varrho^2 & \varrho^3 & \cdots & \varrho^{N-1} \\ \varrho & 1 & \varrho & \varrho^2 & \cdots & \varrho^{N-2} \\ \varrho^2 & \varrho & 1 & \varrho & \cdots & \varrho^{N-3} \\ \varrho^3 & \varrho^2 & \varrho & 1 & \cdots & \varrho^{N-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho^{N-1} & \varrho^{N-2} & \varrho^{N-3} & \varrho^{N-4} & \cdots & 1 \end{bmatrix}$$

➡ KLT Transform matrix only depends on correlation coefficient ϱ and the transform size N

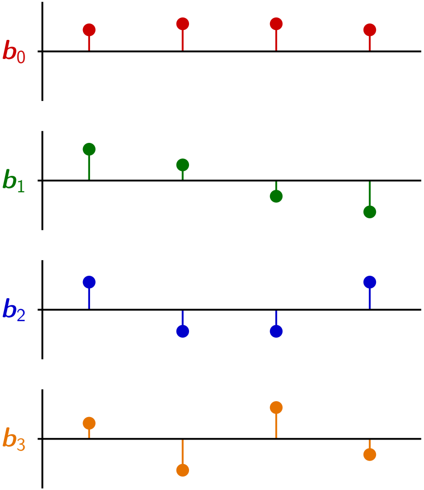
KLT of size $N = 4$ for AR(1) Source with Correlation Coefficient $\varrho = 0.5$

$$\boldsymbol{C}_{SS} = \sigma_S^2 \begin{bmatrix} 1.0000 & 0.5000 & 0.2500 & 0.1250 \\ 0.5000 & 1.0000 & 0.5000 & 0.2500 \\ 0.2500 & 0.5000 & 1.0000 & 0.5000 \\ 0.1250 & 0.2500 & 0.5000 & 1.0000 \end{bmatrix}$$

$$\boldsymbol{A}_{KLT} = \begin{bmatrix} 0.4352 & 0.5573 & 0.5573 & 0.4352 \\ 0.6325 & 0.3162 & -0.3162 & -0.6325 \\ 0.5573 & -0.4352 & -0.4352 & 0.5573 \\ 0.3162 & -0.6325 & 0.6325 & -0.3162 \end{bmatrix}$$

$$\boldsymbol{C}_{UU} = \sigma_S^2 \begin{bmatrix} 2.0856 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0.5394 & 0 \\ 0 & 0 & 0 & 0.3750 \end{bmatrix}$$

$$G_{EC} = 1.2408 \quad (0.94\,\text{dB})$$



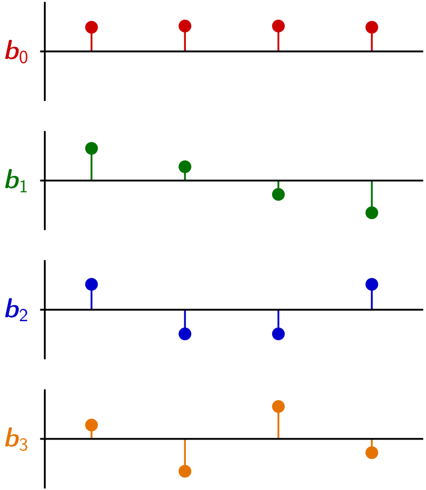
KLT of size $N = 4$ for AR(1) Source with Correlation Coefficient $\varrho = 0.9$

$$\boldsymbol{C}_{SS} = \sigma_S^2 \begin{bmatrix} 1.0000 & 0.9000 & 0.8100 & 0.7290 \\ 0.9000 & 1.0000 & 0.9000 & 0.8100 \\ 0.8100 & 0.9000 & 1.0000 & 0.9000 \\ 0.7290 & 0.8100 & 0.9000 & 1.0000 \end{bmatrix}$$

$$\boldsymbol{A}_{KLT} = \begin{bmatrix} 0.4874 & 0.5123 & 0.5123 & 0.4874 \\ 0.6498 & 0.2789 & -0.2789 & -0.6498 \\ 0.5123 & -0.4874 & -0.4874 & 0.5123 \\ 0.2789 & -0.6498 & 0.6498 & -0.2789 \end{bmatrix}$$

$$\boldsymbol{C}_{UU} = \sigma_S^2 \begin{bmatrix} 3.5266 & 0 & 0 & 0 \\ 0 & 0.3096 & 0 & 0 \\ 0 & 0 & 0.1024 & 0 \\ 0 & 0 & 0 & 0.0614 \end{bmatrix}$$

$$G_{EC} = 3.4746 \quad (5.41\,\mathrm{dB})$$



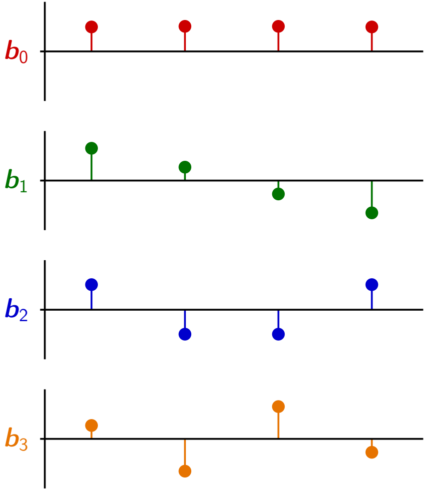
KLT of size $N = 4$ for AR(1) Source with Correlation Coefficient $\varrho = 0.95$

$$\boldsymbol{C}_{SS} = \sigma_S^2 \begin{bmatrix} 1.0000 & 0.9500 & 0.9025 & 0.8574 \\ 0.9500 & 1.0000 & 0.9500 & 0.9025 \\ 0.9025 & 0.9500 & 1.0000 & 0.9500 \\ 0.8574 & 0.9025 & 0.9500 & 1.0000 \end{bmatrix}$$

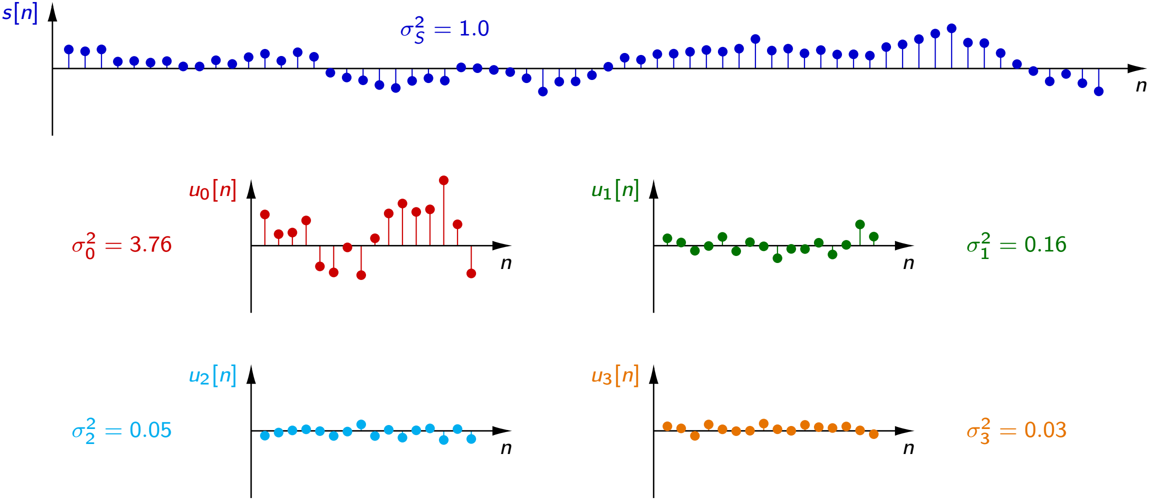
$$\boldsymbol{A}_{KLT} = \begin{bmatrix} 0.4937 & 0.5062 & 0.5062 & 0.4937 \\ 0.6516 & 0.2747 & -0.2747 & -0.6516 \\ 0.5062 & -0.4937 & -0.4937 & 0.5062 \\ 0.2747 & -0.6516 & 0.6516 & -0.2747 \end{bmatrix}$$

$$\boldsymbol{C}_{UU} = \sigma_S^2 \begin{bmatrix} 3.7568 & 0 & 0 & 0 \\ 0 & 0.1627 & 0 & 0 \\ 0 & 0 & 0.0506 & 0 \\ 0 & 0 & 0 & 0.0300 \end{bmatrix}$$

$$G_{EC} = 5.7307 \quad (7.58\,\text{dB})$$



Gauss-Markov with $\varrho = 0.95$: KLT of size $N = 4$



Optimality of KLT

Properties of KLT

- KLT produces uncorrelated transform coefficients
- KLT minimizes geometric mean of transform coefficient variances $\tilde{\sigma}^2$ (diagonal elements of $\boldsymbol{C}_{UU}$)
- KLT achieves maximum possible energy compaction $G_{EC} = \sigma_S^2/\tilde{\sigma}^2$

Gaussian Sources

- Obviously, KLT maximizes high-rate transform gain $G_T = G_{EC}$
- More general: **For Gaussian sources and MSE distortion, the KLT is the optimal orthogonal transform**
  - ➡ Valid for all possible rate allocations (including the optimal one)
  - ➡ Proof can be found in [Goyal, 2000] or [Wiegand, Schwarz, 2011]

Non-Gaussian Sources

- Other transforms may yield a better coding efficiency
- For most sources, KLT still provides good coding efficiency

KLT for Gauss-Markov: Geometric Mean of Variances

- High-rate distortion-rate function for KLT of size N

$$D(R) = \varepsilon^2 \cdot \tilde{\sigma} \cdot 2^{-2R} = \varepsilon^2 \cdot \tilde{\xi} \cdot 2^{-2R}$$

- Linear algebra: Product of eigenvalues = determinant

$$\tilde{\xi} = \left(\prod_{k=0}^{N-1} \xi_k \right)^{\frac{1}{N}} = |\boldsymbol{C}_N|^{\frac{1}{N}}$$

- Determinant of Gauss-Markov source (or general AR(1) sources)

$$|\boldsymbol{C}_N| = \begin{vmatrix} \sigma_S^2 & \varrho \cdot \sigma_S^2 & \varrho^2 \cdot \sigma_S^2 & \cdots & \varrho^{N-1} \cdot \sigma_S^2 \\ \varrho \cdot \sigma_S^2 & \sigma_S^2 & \varrho \cdot \sigma_S^2 & \cdots & \varrho^{N-2} \cdot \sigma_S^2 \\ \varrho^2 \cdot \sigma_S^2 & \varrho \cdot \sigma_S^2 & \sigma_S^2 & \cdots & \varrho^{N-3} \cdot \sigma_S^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho^{N-1} \cdot \sigma_S^2 & \varrho^{N-2} \cdot \sigma_S^2 & \varrho^{N-3} \cdot \sigma_S^2 & \cdots & \sigma_S^2 \end{vmatrix}$$

KLT for Gauss-Markov: Laplace Expansion of Determinant

- Expand determinant along the first column using Laplace's formula

$$|\boldsymbol{C}_N| = \sum_{k=0}^{N-1} (-1)^k \, c_{k,0} \left| \boldsymbol{C}_N^{(k,0)} \right| = \sum_{k=0}^{N-1} (-1)^k \, \sigma_S^2 \, \varrho^k \left| \boldsymbol{C}_N^{(k,0)} \right|$$

with $c_{k,\ell}$ being the element at row k and column ℓ, and

$\boldsymbol{C}_N^{(k,\ell)}$ being the matrix that is obtained by removing the k-th row and ℓ-th column from $\boldsymbol{C}_N$

- Consider matrices $\boldsymbol{C}_N^{(k,0)}$ with $k > 1$
  - First row is equal to second row multiplied by ϱ
  - First row is linearly dependent of second row and, hence, we have

$$\forall k > 1, \qquad \left| \boldsymbol{C}_N^{(k,0)} \right| = 0$$

➞ Above formula simplifies to

$$|\boldsymbol{C}_N| = \sigma_S^2 \left| \boldsymbol{C}_N^{(0,0)} \right| - \sigma_S^2 \, \varrho \left| \boldsymbol{C}_N^{(1,0)} \right|$$

KLT for Gauss-Markov: Determinants of Sub-Matrices

- Matrix $\boldsymbol{C}_N^{(0,0)}$ is equal to $\boldsymbol{C}_{N-1}$
- Matrix $\boldsymbol{C}_N^{(1,0)}$ has the form

$$\boldsymbol{C}_N^{(1,0)} = \begin{vmatrix} \varrho \cdot \sigma_S^2 & \varrho^2 \cdot \sigma_S^2 & \varrho^3 \cdot \sigma_S^2 & \cdots & \varrho^{N-1} \cdot \sigma_S^2 \\ \varrho \cdot \sigma_S^2 & \sigma_S^2 & \varrho \cdot \sigma_S^2 & \cdots & \varrho^{N-3} \cdot \sigma_S^2 \\ \varrho^2 \cdot \sigma_S^2 & \varrho \cdot \sigma_S^2 & \sigma_S^2 & \cdots & \varrho^{N-4} \cdot \sigma_S^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varrho^{N-2} \cdot \sigma_S^2 & \varrho^{N-3} \cdot \sigma_S^2 & \varrho^{N-4} \cdot \sigma_S^2 & \cdots & \sigma_S^2 \end{vmatrix}$$

  ➡ Same as $\boldsymbol{C}_{N-1}$ except that first row is multiplied by ϱ

  ➡ Determinant is given by $|\boldsymbol{C}_N^{(1,0)}| = \varrho \, |\boldsymbol{C}_{N-1}|$

➡ Recursive formula for $|\boldsymbol{C}_N|$

$$\begin{aligned} |\boldsymbol{C}_N| &= \sigma_S^2 \, \big|\boldsymbol{C}_N^{(0,0)}\big| - \sigma_S^2 \, \varrho \, \big|\boldsymbol{C}_N^{(1,0)}\big| = \sigma_S^2 \, |\boldsymbol{C}_{N-1}| - \sigma_S^2 \, \varrho \cdot \varrho \, |\boldsymbol{C}_{N-1}| \\ &= \sigma_S^2 \, (1 - \varrho^2) \, |\boldsymbol{C}_{N-1}| \end{aligned}$$

KLT for Gauss-Markov: High-Rate Distortion-Rate Function

- Formula for determinant of auto-covariance matrix $|\boldsymbol{C}_N|$

$$
\begin{aligned}
|\boldsymbol{C}_N| &= \sigma_S^2\,(1-\varrho^2)\,|\boldsymbol{C}_{N-1}| \\
&= \left(\sigma_S^2\,(1-\varrho^2)\right)^{N-1}\cdot|\boldsymbol{C}_1| \qquad \left(\text{note: } \boldsymbol{C}_1 = \left[\,\sigma_S^2\,\right]\right) \\
&= \sigma_S^{2N}\cdot(1-\varrho^2)^{N-1}
\end{aligned}
$$

➡ Geometric mean of transform coefficient variances

$$\tilde{\sigma}^2 = |\boldsymbol{C}_N|^{\frac{1}{N}} = \sigma_S^2\cdot(1-\varrho^2)^{\frac{N-1}{N}}$$

➡ High-rate distortion-rate function for KLT of size N

$$D_{KLT}^N(R) = \varepsilon^2\cdot\sigma_S^2\cdot(1-\varrho^2)^{\frac{N-1}{N}}\cdot 2^{-2R} \qquad \left(\text{ECSQ: } \varepsilon^2 = {}^{\pi e}\!/_{6}\right)$$

➡ High-rate transform coding gain for KLT of size N

$$G_{KLT}^N = \frac{D_{SC}(R)}{D_{KLT}^N(R)} = \frac{\varepsilon^2\cdot\sigma_S^2\cdot 2^{-2R}}{\varepsilon^2\cdot\sigma_S^2\cdot(1-\varrho^2)^{\frac{N-1}{N}}\cdot 2^{-2R}} = \left(\frac{1}{1-\varrho^2}\right)^{\frac{N-1}{N}}$$

KLT + ECSQ for Gauss-Markov at High Rates: Asymptotic Limits

Comparison to Rate-Distortion Bound

- Combination of KLT and ECSQ: Distortion increase relative to Shannon lower bound $D_L(R)$

$$\frac{D^N_{KLT}(R)}{D_L(R)} = \frac{\frac{\pi e}{6} \cdot \sigma_S^2 \cdot (1-\varrho^2)^{\frac{N-1}{N}} \cdot 2^{-2R}}{\sigma_S^2 \cdot (1-\varrho^2) \cdot 2^{-2R}} = \frac{\pi e}{6} \cdot \left(\frac{1}{1-\varrho^2} \right)^{\frac{1}{N}}$$

Asymptotic Limits for Large Transforms ($N \to \infty$)

➡ Transform coding gain

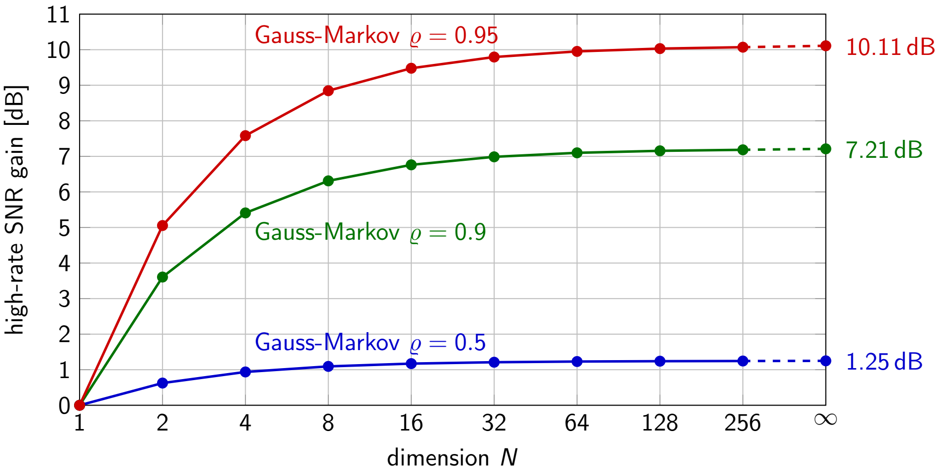
$$G^{\infty}_{KLT} = \lim_{N\to\infty} \left(\frac{1}{1-\varrho^2} \right)^{\frac{N-1}{N}} = \frac{1}{1-\varrho^2}$$

➡ Distortion-rate function for KLT and ECSQ

$$D^{\infty}_{KLT}(R) = \frac{\pi e}{6} \cdot \sigma_S^2 \cdot (1-\varrho^2) \cdot 2^{-2R} \quad \Rightarrow \quad \frac{D^{\infty}_{KLT}(R)}{D_L(R)} = \frac{\pi e}{6} \approx 1.42 \quad (1.53\,\text{dB})$$

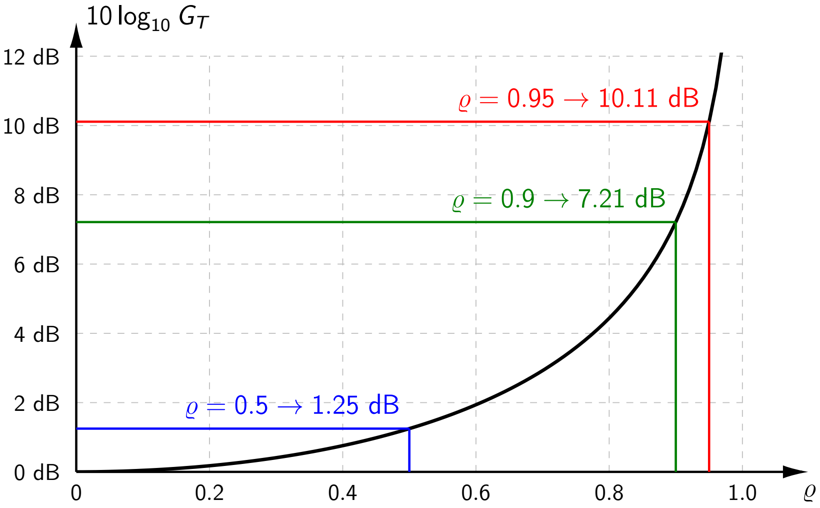
➡ Gap to rate-distortion bound reduces to space-filling advantage of VQ

KLT for Gauss-Markov: High-Rate Transform Coding Gain



➡ **Identical to memory advantage of unconstrained vector quantization !**

Asymptotic Transform Gain for Gauss-Markov Processes



Coding Experiment: KLT Coding of Gauss-Markov ($\varrho = 0.9$)

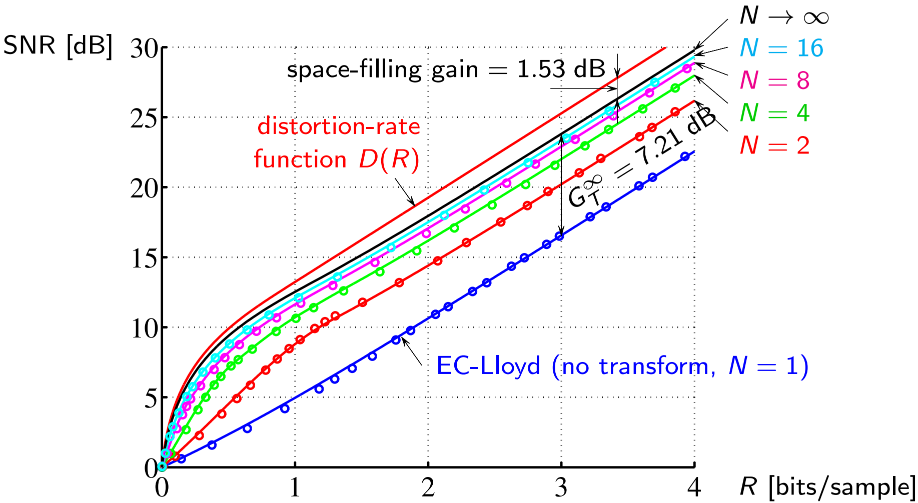


Image Coding: 2D Transform

Image Coding

- Statistical dependencies in multiple directions
  (e.g., between vertically and horizontally adjacent samples)

➡ Images are typically coded using $N \times M$ blocks of samples

Straightforward Extension to Two Dimensions

- Arrange samples of $N \times M$ block into vector of size NM

$$\boldsymbol{s}_{blk} = \begin{bmatrix} s_{00} & s_{01} & s_{02} \\ s_{10} & s_{11} & s_{12} \\ s_{20} & s_{21} & s_{22} \\ s_{30} & s_{31} & s_{32} \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{s}_{vec} = \begin{bmatrix} s_{00} & s_{01} & s_{02} & s_{10} & s_{11} & s_{12} & s_{20} & s_{21} & s_{22} & s_{30} & s_{31} & s_{32} \end{bmatrix}^{\mathrm{T}}$$

- Design transform matrix $\boldsymbol{A}$ for vectors $\boldsymbol{s}_{vec}$ of size NM

➡ Transform matrix has the size $(NM) \times (NM)$

Image Coding: Separable 2D Transform

Separable 2D Transform

- Successive 1D transform for rows and columns of an $N \times M$ image block

➞ Separable orthogonal transform

$$\begin{bmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \\ u_{30} & u_{31} & u_{32} \end{bmatrix} = \boldsymbol{A}_{ver} \cdot \begin{bmatrix} s_{00} & s_{01} & s_{02} \\ s_{10} & s_{11} & s_{12} \\ s_{20} & s_{21} & s_{22} \\ s_{30} & s_{31} & s_{32} \end{bmatrix} \cdot \boldsymbol{A}_{hor}^{\mathrm{T}}$$

with $\boldsymbol{A}_{ver}$ being an $N \times N$ transform matrix for transforming the columns, and
$\boldsymbol{A}_{hor}$ being an $M \times M$ transform matrix for transforming the rows

- Inverse transform is also separable

$$\boldsymbol{s}' = \boldsymbol{A}_{ver}^{\mathrm{T}} \cdot \boldsymbol{u}' \cdot \boldsymbol{A}_{hor}$$

- Independent design of horizontal and vertical transform matrix

➞ Great importance: Significant reduction in complexity

Image Example: Comparison of Separable and Non-Separable 2D KLT

original

4×4 **non-separable KLT**

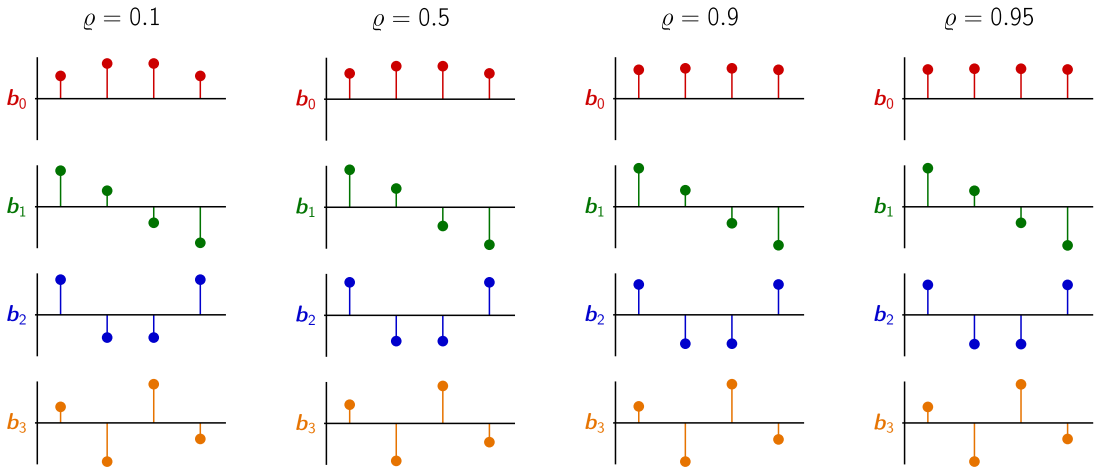
$G_{EC} = 23.804$ dB

4×4 **separable KLT**

$G_{EC} = 23.635$ dB

- ➡ Energy compaction gain decreases by 0.17 dB due to usage of separable transform
- ➡ Corresponds to distortion increase of about 1.04 (at same rate)

Convergence of KLT for AR(1) Sources



➡ KLT transform matrix converges for $\varrho \to 1$

The Discrete Cosine Transform (DCT)

Transform Matrix of the Discrete Cosine Transform (DCT)

- The DCT is an orthogonal transform
- The transform matrix $\boldsymbol{A}_{DCT} = \{a_{kn}\}$ has the elements

$$a_{kn} = \alpha_k \cdot \cos\left(\frac{\pi}{N}\, k \left(n + \frac{1}{2} \right) \right) \qquad \text{with} \qquad \alpha_k = \begin{cases} \sqrt{1/N} & : \quad k = 0 \\ \sqrt{2/N} & : \quad k \neq 0 \end{cases}$$

- The basis vectors $\boldsymbol{b}_k = \{a_{kn}\}$ represent sampled cosine functions of different frequencies

Relation to KLT

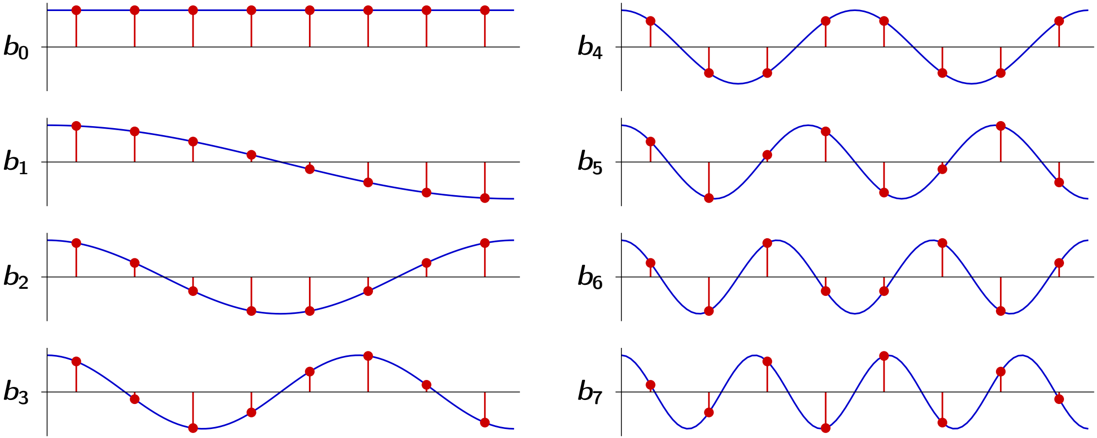
- Unit-norm eigenvectors of $\boldsymbol{C}_{SS}$ approach DCT basis vectors for $\varrho \to 1$

Advantages of DCT

- Transform matrix does not depends on the input signal
- Fast algorithms for computing the forward and inverse transforms

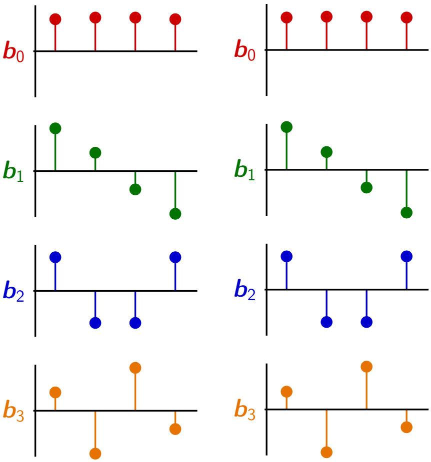
Basis Functions of the DCT (Example for $N = 8$)

$$\boldsymbol{b_k}[n] = \alpha_k \cdot \cos\left(\frac{\pi}{8}\, k \left(n + \frac{1}{2} \right) \right)$$



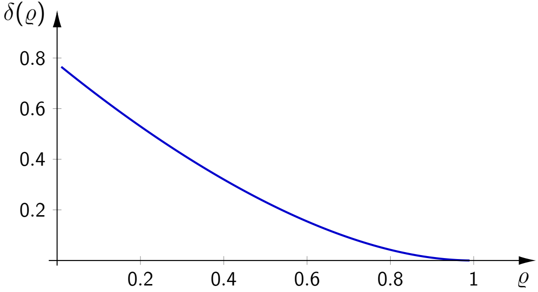
AR(1) Sources: KLT Convergence Towards DCT for $\varrho \to 1$

KLT for $\varrho = 0.9$ DCT



difference between transform matrices of KLT and DCT for $N = 8$

$$\delta(\varrho) = \| \boldsymbol{A}_{KLT}(\varrho) - \boldsymbol{A}_{DCT} \|_2^2$$



AR(1) Sources: Energy Compaction of KLT and DCT for $N = 8$

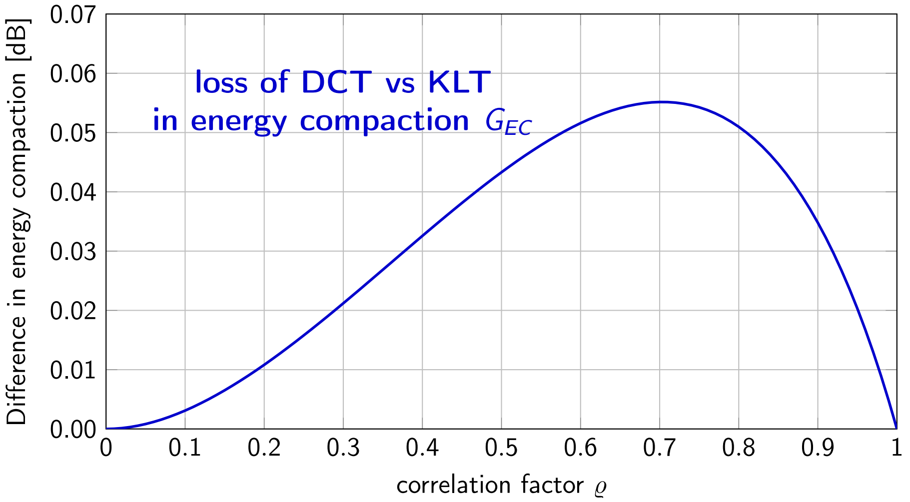


Image Example: Comparison of 2D DCT and Separable 2D KLT

original

4×4 **separable KLT**

$G_{EC} = 23.6350$ dB

4×4 **separable DCT**

$G_{EC} = 23.6285$ dB

➡ Energy compaction gain decreases by 0.0065 dB due to usage of DCT instead of separable KLT

➡ Corresponds to distortion increase of about 1.0015 (at same rate)

Summary of Lecture

Karhunen Loève Transform (KLT)

- Orthogonal transform that produces uncorrelated transform coefficients
- Basis vectors are the unit-norm eigenvectors of auto-covariance matrix
- Minimizes geometric mean of transform coefficients, maximizes energy compaction
- Optimal transform for Gaussian sources

Discrete Cosine Transform (DCT)

- Signal independent orthogonal transform
- Basis vectors: Samples cosine functions of different frequencies
- KLT for AR(1) approaches DCT for $\varrho \rightarrow 1$
- Typical: Energy compaction very close to that of KLT

2D Transforms

- Separable transforms for reducing implementation complexity
- Typically, small loss versus non-separable KLT

Exercise 1: Transform Coding Gain for Gauss-Markov Sources

In the video coding standard ITU-T Rec. H.264 the following forward transform is used:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ 1 & -1 & -1 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix}$$

1. How large is the high-rate transform coding gain (in dB) for a zero-mean Gauss-Markov process with the correlation factor $\rho = 0.9$?
2. By what amount (in dB) can the high-rate transform coding gain be increased if the transform is replaced by a KLT?

NOTE: The basis functions of the given transform are orthogonal to each other, but they don't have the same norm. This has to be taken into account in the calculations.

Exercise 2: High-Rate Bit Allocation for KLT

Consider a zero-mean Gauss-Markov process with variance $\sigma_S^2 = 1$ and correlation coefficient $\rho = 0.9$.
As transform a KLT of size 3 is used, the resulting transform coefficient variances are

$$\sigma_0^2 = 2.7407, \qquad \sigma_1^2 = 0.1900, \qquad \sigma_2^2 = 0.0693$$

Consider high-rate quantization with optimal entropy-constrained scalar quantizers.

1. Derive the high-rate operational distortion rate function.
2. What is the optimal high-rate bit allocation scheme for a given overall rate R?
3. Determine the component rates, the overall distortion, and the SNR for a given overall bit rate R of 4 bit per sample.
4. Determine the high-rate transform coding gain.

Exercise 3: Transform of Image Blocks using the DCT (Implementation)

Prepare a lossy image codec for PPM images. Implement the following:

1. **Reading and writing of PPM images**
   - For details on the PPM format, see older exercises
   - Re-use code from older exercises (see KVV)

2. **Transform coding for sample blocks**
   - a) Apply a separable 8×8 DCT for an image block (or make the block size $N \times N$ variable)
   - b) Quantize the resulting transform coefficient by simple rounding (using a fixed quantization step size)
   - c) Reconstruct transform coefficients (multiplication with quantization step size)
   - d) Apply the inverse transform (inverse DCT)

3. **Test Your Implementation**
   - Apply the transform coding to all sample blocks of an image (without writing a bitstream)
   - ➡ Test the transforms without quantization
   - ➡ Test the transform coding with different quantization step sizes (look at reconstructed images)