Exercise 1: Properties of Expected Values

Proof the following properties of expected values

- Linearity

$$\mathrm{E}\{\, a\,X + b\,Y \,\} = a\,\mathrm{E}\{\, X \,\} + b\,\mathrm{E}\{\, Y \,\}$$

- For two independent random variables X and Y, we have

$$\mathrm{E}\{\, XY \,\} = \mathrm{E}\{\, X \,\}\, \mathrm{E}\{\, Y \,\}$$

- Iterative expectation rule

$$\mathrm{E}\{\, \mathrm{E}\{\, g(X) \,|\, Y \,\}\,\} = \mathrm{E}\{\, g(X) \,\}$$

Exercise 2: Correlation and Independence

Investigate the relationship between independence and correlation.

> Two random variables X and Y are said to be *correlated* if and only if their covariance $\sigma_{XY}^2 = \mathrm{E}\{\,(X - \mathrm{E}\{\,X\,\})(Y - \mathrm{E}\{\,Y\,\})\,\}$ is not equal to 0.

(a) Can two independent random variables X and Y be correlated?

(b) Are two uncorrelated random variables X and Y also independent?

Exercise 3: Marginal Pmf of Markov Process (Optional)

Given is a stationary discrete Markov process with the alphabet $\mathcal{A} = \{a, b, c\}$ and the conditional pmf

$$p(x_k \,|\, x_{k-1}) = \mathrm{P}(X_k = x_k \,|\, X_{k-1} = x_{k-1})$$

listed in the table below

x_n	$p(x_n \,|\, a)$	$p(x_n \,|\, b)$	$p(x_n \,|\, c)$	$p(x_n)$
a	0.90	0.15	0.25	?
b	0.05	0.80	0.15	?
c	0.05	0.05	0.60	?

➡ Determine the marginal pmf $p(x) = \mathrm{P}(X_k = x)$.

Exercise 4: Unqiue Decodability

Given is a discrete iid process $\mathbf{X}$ with the alphabet $\mathcal{A} = \{a, b, c, d, e, f, g\}$.
The pmf $p_X(x)$ and five example codes are listed in the following table.

x	$p_X(x)$	A	B	C	D	E
a	1/3	1	0	00	01	1
b	1/9	0001	10	010	101	100
c	1/27	000000	110	0110	111	100000
d	1/27	00001	1110	0111	010	10000
e	1/27	000001	11110	100	110	000000
f	1/9	001	111110	101	100	1000
g	1/3	01	111111	11	00	10

(a) Calculate the entropy of the source.

(b) Calculate the average codeword lengths and the redundancies for the given codes.

(c) Which of the given codes are uniquely decodable codes?

Exercise 5: Prefix Codes

Given is a random variable X with the alphabet $\mathcal{A}_X = \{a, b, c, d, e, f\}$.

Two sets of codeword lengths are given in the following table.

letter	set A	set B
a	2	1
b	2	3
c	2	3
d	3	3
e	3	4
f	4	4

(a) For which set(s) can we construct a uniquely decodable code?

(b) Develop a prefix code for the set(s) determined in **(a)**.

(c) Consider the prefix code(s) developed in **(b)**. Is it possible to find a pmf p for which the developed code yields an average codedword length $\bar{\ell}$ equal to the entropy $H(p)$? If yes, write down the probability masses.

Exercise 6: Maximum Entropy (Optional)

Consider an iid process with an alphabet of size N (i.e., the alphabet includes N different letters).

(a) Calculate the entropy H_{uni} for the case that the pmf represents a uniform pmf:

$$\forall k, \quad p_k = \frac{1}{N}$$

(b) Show that for all other pmfs (i.e., all non-uniform pmfs), the entropy H is less than H_{uni}.