Exercise 1: Covariance Function for AR(1) Sources

Given is a zero-mean iid process $\boldsymbol{Z} = \{Z_n\}$ with variance σ_Z^2. For a given correlation coefficient ϱ and mean μ_S, a stationary continuous AR(1) process is constructed according to

$$S_n = \mu_S + \varrho\,(S_{n-1} - \mu_S) + Z_n$$

(a) What is the variance σ_S^2 of the resulting process $\{S_n\}$?

(b) How do we have to modify the construction rule in order to get an AR(1) process with a pre-defined variance σ_S^2?

(c) Proof that

$$\operatorname{cov}(S_k, S_\ell) = \sigma_S^2 \cdot \varrho^{|k-\ell|}$$

Exercise 2: Mutual Information for Discrete Case

Given is a stationary Markov process $\boldsymbol{S} = \{S_n\}$ with the binary symbol alphabet $\mathcal{A} = \{x, y\}$. The conditional symbol probabilities $p(s_n \,|\, s_{n-1})$ are given in the table below.

s_n	$p(s_n \,\vert\, s_{n-1} = x)$	$p(s_n \,\vert\, s_{n-1} = y)$
x	$3/4$	$1/4$
y	$1/4$	$3/4$

Calculate:

- the marginal entropy $H(S_n)$,
- the joint entropy $H(S_n, S_{n+1})$ for two successive random variables,
- the conditional entropy $H(S_n \,|\, S_{n-1})$ for a random variable given the preceding random variable,
- the mutual information $I(S_n; S_{n+1})$ between two successive random variables.

Exercise 3: Mutual Information for Stationary Gauss-Markov (Optional)

Consider a stationary Gauss-Markov process $\boldsymbol{X} = \{X_n\}$ with mean μ, variance σ^2, and the correlation coefficient ϱ (correlation coefficient between two successive random variables).

Determine the mutual information $I(X_k; X_{k+N})$ between two random variables X_k and X_{k+N}, where the distance between the random variables is N times the sampling interval.

Interpret the results for the special cases $\varrho = -1$, $\varrho = 0$, and $\varrho = 1$.

Hint: It can be shown that

$$E\left\{(\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}} \cdot \boldsymbol{C}_N^{-1} \cdot (\boldsymbol{X} - \boldsymbol{\mu})\right\} = N,$$

which can be useful for the problem.

Exercise 4: Shannon Lower Bound (MSE Distortion)

Determine the Shannon lower bound for MSE distortion, as distortion-rate function, for iid processes with the following pdfs:

- The exponential pdf $f_E(x) = \lambda \cdot e^{-\lambda \cdot x}$, with $x \geq 0$
- The half-normal pdf $f_H(x) = \sqrt{\frac{4a}{\pi}} \cdot e^{-a \cdot x^2}$, with $x \geq 0$

Express the distortion-rate functions for the Shannon lower bound as a function of the variance σ^2.

Which of the given pdfs is easier to code (if the variance is the same)?

Verify that both pdfs are easier to code than the Gaussian iid with the same variance.

Exercise 5: Shannon Lower Bound for MAE Distortion (Optional)

Consider rate-distortion bounds for MAE (mean absolute error) distortion.

- Calculate the differential entropy for the Laplace pdf

$$f(s) = \frac{\lambda}{2} \cdot e^{-\lambda \cdot |x-\mu|}$$

  as a function of $m = E\{|X - \mu|\}$.

- Show that the Laplace pdf is the pdf with the maximum differential entropy of all pdfs with the same value of $m = E\{|X - \mu|\}$.

- Derive the Shannon lower bound for iid sources and the MAE distortion measure $d(x, x') = |x - x'|$. Formulate the Shannon lower bound as rate-distortion and as distortion-rate function.

- Calculate the Shannon lower bound for the MAE distortion measure for the Laplace iid source.