Exercise 1: Implement the Entropy-Constrained Lloyd Algorithm (optional)

Implement the entropy-constrained Lloyd algorithm using a programming language of your choice.

- Test the algorithm for
  - a unit-variance Gaussian pdf:
    $$f(s) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\, s^2}$$
  - a unit-variance Laplacian pdf:
    $$f(s) = \frac{1}{\sqrt{2}}\, e^{-\sqrt{2}\,|s|}$$
- Use the following Lagrange multipliers: $\lambda = 0.5, 0.2, 0.1, 0.05, 0.02, 0.01$.
- Determine the rate (entropy) R and the distortion D for your quantizers.
- Compare the R-D performance of your quantizers to the high-rate approximation.

You can implement the EC Lloyd algorithm that directly uses the pdf or the EC Lloyd algorithm that uses a training set (files with 1 000 000 samples in float32 format are provided on the course web site)

Exercise 2: Quantization of Sources with Memory

Consider a discrete Markov process $\mathbf{X} = \{X_n\}$ with the symbol alphabet $\mathcal{A}_X = \{0, 2, 4, 6\}$ and the conditional pmf

$$p_{X_n|X_{n-1}}(x_n|x_{n-1}) = \begin{cases} a & : \quad x_n = x_{n-1} \\ \frac{1}{3}(1-a) & : \quad x_n \neq x_{n-1} \end{cases}$$

The parameter a, with $0 < a < 1$, is a variable that specifies the probability that the current symbol is equal to the previous symbol. For $a = 1/4$, our source $\mathbf{X}$ would be i.i.d.

Given is a two-interval quantizer with the reconstruction levels $s'_0 = 1$ and $s'_1 = 5$ and the decision threshold $u_1 = 3$.

(a) Assume optimal entropy coding using the marginal probabilities of the quantization indices and determine the rate-distortion point of the quantizer.

(b) Can the overall quantizer performance be improved by applying conditional entropy coding (e.g., using arithmetic coding with conditional probabilities)?

How does it depend on the parameter a?

Exercise 3: High-Rate Quantization

Consider scalar quantization of a Laplacian source at high rates:

$$f(x) = \frac{\lambda}{2} \cdot e^{-\lambda\,|x|} \qquad \text{with} \qquad \sigma_S^2 = \frac{2}{\lambda^2}$$

In a given system, the used quantizer is a Lloyd quantizer with fixed-length entropy coding (the number of quantization intervals represents a power of 2).

How many bits per sample (for the same MSE distortion) can be saved if the quantizer is replaced by an entropy-constrained quantizer with optimal entropy coding?

Note:

Assume that the operation points of the quantizers can be accurately described by the corresponding high rate approximations.

Exercise 4: Implementation of First Lossy Image Codec

- Use the PPM format as raw data format (see earlier exercise on lossless image coding)
- Use any of the lossless image codecs available in the KVV (or your own implementation) as basis

Implement an Image Encoder

- Quantize the original image samples $s[x, y]$ using a fixed quantization step size Δ
  - ➡ Simple rounding is sufficient for our purpose: $k[x, y] = \text{round}(\, s[x, y]/\Delta\,)$
  - ➡ Transmit the quantization step size Δ at the beginning of the bitstream
- Use the lossless codec for coding the quantization indexes $k[x, y]$

Implement the corresponding Image Decoder

- Decode the quantization indexes $k[x, y]$ using the lossless codec
- Reconstruct the image samples according to: $s'[x, y] = k[x, y] \cdot \Delta$

Test your Codec

- Code selected test images with different quantization step sizes (e.g., $\Delta = 2, 4, 8, 16, 32, 64$)
- Measure the compression factors (based on the file sizes) and judge the image quality by visual inspection

Exercise 5: Quantization of Exponential Source (optional / more difficult)

Consider uniform threshold quantization of an exponential pdf given by $f(x) = a\, e^{-ax}$.

With Δ denoting the quantization step size, the thresholds are given by $u_k = k\Delta$, with $k = 0, 1, 2, \cdots$.

(a) Determine the pmf for the quantization indexes.

Calculate the rate (entropy) as function of the probability $p = P(X > \Delta) = e^{-a\Delta}$.

Describe an entropy coding scheme for the quantization indices that virtually achieves the entropy.

(b) Derive a formula for the optimal reconstruction levels s'_k, for MSE distortion, as function of the quantization step size Δ, the lower interval boundaries u_k, and the probability $p = e^{-a\Delta}$.

(c) Is the obtained quantizer an optimal entropy-constrained scalar quantizer?

(d) Determine the distortion in dependence of the quantization step size for the developed quantizer.

Hint: For $|a| < 1$,

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \qquad \sum_{k=0}^{\infty} k\, a^k = \frac{a}{(1-a)^2}, \qquad \sum_{k=0}^{\infty} k^2\, a^k = \frac{a\,(1+a)}{(1-a)^3}$$