Exercise 1

Consider a discrete Markov process $\mathbf{X} = \{X_n\}$ with the symbol alphabet $\mathcal{A}_X = \{0, 2, 4, 6\}$ and the conditional pmf

$$p_{X_n|X_{n-1}}(x_n|x_{n-1}) = \begin{cases} a & : \quad x_n = x_{n-1} \\ \frac{1}{3}(1-a) & : \quad x_n \neq x_{n-1} \end{cases},$$

for $x_n, x_{n-1} \in \mathcal{A}_X$. The parameter a, with $0 < a < 1$, is a variable that specifies the probability that the current symbol is equal to the previous symbol. For $a = 1/4$, our source $\mathbf{X}$ would be i.i.d.

Given is a quantizer of size 2 with the reconstruction levels $s'_0 = 1$ and $s'_1 = 5$ and the decision threshold $u_1 = 3$.

(a) Assume optimal entropy coding using the marginal probabilities of the quantization indices and determine the rate-distortion point of the quantizer.

(b) Can the overall quantizer performance be improved by applying conditional entropy coding (e.g., using arithmetic coding with conditional probabilities)? How does it depend on the parameter a?

Exercise 2 (Optional) – Part 1

Consider uniform threshold quantization (i.e., quantization with fixed-size quantization intervals) of an exponential pdf given by $f(x) = a\,e^{-ax}$.
With Δ denoting the quantization step size, the decision thresholds are given by $u_k = k\Delta$, with $k = 0, 1, 2, \cdots$.

(a) Determine the probability mass function of the quantization indexes and calculate the rate (assuming optimal entropy coding) as function of the probability $p = P(X > \Delta) = e^{-a\Delta}$.

Describe an entropy coding scheme for the quantization indices that virtually achieves the entropy.

Hint: For $|a| < 1$, $\displaystyle\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$ and $\displaystyle\sum_{k=0}^{\infty} k\,a^k = \frac{a}{(1-a)^2}$

(b) Derive a formula for the optimal reconstruction levels s'_k, for MSE distortion, as function of the quantization step size Δ and as function of the lower interval borders u_k and the probability $p = e^{-a\Delta}$.

Exercise 2 (Optional) – Part 2

(c) Is the obtained quantizer an optimal entropy-constrained scalar quantizer?

What is the relationship between the Lagrange parameter λ and the quantization step size Δ? And what is the relationship between the Lagrange parameter λ and probability $p = e^{-a\Delta}$?

(d) Determine the distortion in dependence of the quantization step size for the developed quantizer.

$$\text{Hint: For } |a| < 1, \qquad \sum_{k=0}^{\infty} k^2\, a^k = \frac{a\,(1+a)}{(1-a)^3}$$

(e) Compare the operational rate-distortion function (using a parametric formulation) to the Shannon lower bound.

Plot the operational rate-distortion function and the Shannon lower bound in one diagram for rates from 0 to 5 bit per sample.

Plot the ratio of the distortion for the operational rate-distortion function and the Shannon lower bound for rates from 0 to 5 bit per sample.

Alternative Exercise

Implement the entropy-constrained Lloyd algorithm (either using the pdf or using a large training set) using a programming language of your choice.

- Choose a rather large number of quantization intervals ($K \geq 100$).
- Test the algorithm for different values of λ for
  - a unit-variance Gaussian pdf:

$$f(s) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\, s^2}$$

  - a unit-variance Laplacian pdf:

$$f(s) = \frac{1}{\sqrt{2}}\, e^{-\sqrt{2}\,|s|}$$

- Determine the distortion D and rate $R = H(S')$ for your quantizer designs.
- Compare the R-D performance of your quantizer designs to
  - the Shannon lower bound and
  - the high-rate approximation (Gish and Pierce) for EC-Lloyd quantizers