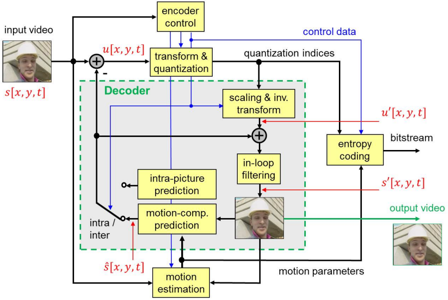
Source Coding and Compression

Heiko Schwarz



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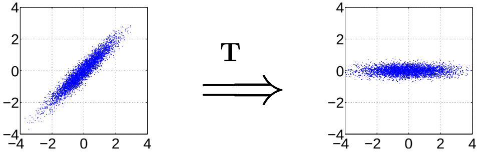
Transform Coding



Outline

Part I: Source Coding Fundamentals

- Probability, Random Variables and Random Processes
- Lossless Source Coding
- Rate-Distortion Theory
- Quantization
- Predictive Coding
- **Transform Coding**
  - Structure of Transform Coding Systems
  - Orthogonal Block Transforms
  - Bit Allocation for Transform Coefficients
  - Karhunen Loéve Transform (KLT)
  - Signal Independent Transforms (Hadamard, FFT, DCT)

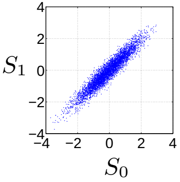
Part II: Application in Image and Video Coding

- Still Image Coding / Intra-Picture Coding
- Hybrid Video Coding (From MPEG-2 Video to H.265/HEVC)

Transform Coding – Introduction

- Another concept for partially exploiting the memory gain
- Used in virtually all lossy image and video coding applications
- Samples of source s are grouped into vectors $\boldsymbol{s}$ of adjacent samples
- Transform coding consists of the following steps
  1. Linear analysis transform $\boldsymbol{A}$, converting source vectors $\boldsymbol{s}$ into transform coefficient vectors $\boldsymbol{u} = \boldsymbol{A}\boldsymbol{s}$
  2. Scalar quantization of the transform coefficients $\boldsymbol{u} \mapsto \boldsymbol{u}'$
  3. Linear synthesis transform $\boldsymbol{B}$, converting quantized transform coefficient vectors $\boldsymbol{u}'$ into decoded source vectors $\boldsymbol{s}' = \boldsymbol{B}\boldsymbol{u}'$

Adjacent Samples

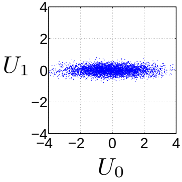


2D Transform:
Rotation by $\varphi = 45^\circ$

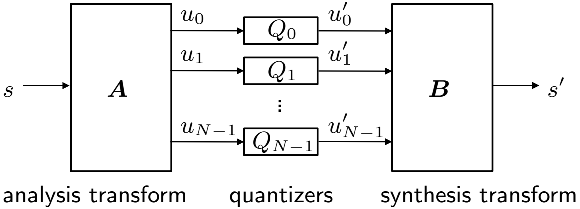
$$\boldsymbol{A} = \begin{bmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{bmatrix}$$

$$\Longrightarrow$$

Transform Coefficients



Structure of Transform Coding Systems



- Synthesis transform is typically inverse of analysis transform
- Separate scalar quantizer Q_n for each transform coefficient u_n
- Vector quantization of all bands or some of them is also possible, but
  - Transforms are designed to have a decorrelating effect (memory gain)
  - Shape gain can be obtained by ECSQ
  - Space-filling gain is left as a possible additional gain for VQ

Combination of decorrelating transformation, scalar quantization and entropy coding is highly efficient – in terms of rate-distortion performance and complexity

Motivation of Transform Coding

Exploitation of statistical dependencies

- Transform are typically designed in a way that, for typical input signals, the signal energy is concentrated in a few transform coefficients
- Coding of a few coefficients and many zero-valued coefficients can be very efficient (e.g., using arithmetic coding, run-length coding)
- Scalar quantization is more effective in transform domain

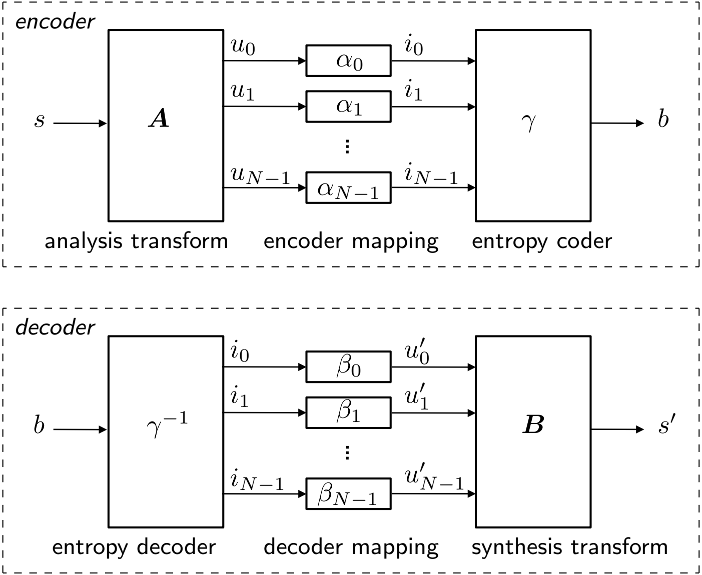
Efficient trade-off between coding efficiency & complexity

- Vector Quantization: Searching through codebook for best matching vector
- Combination of transform and scalar quantization typically results in a substantial reduction in computational complexity

Suitable for quantization using perceptual criteria

- In image & video coding, quantization in transform domain typically leads to an improvement in subjective quality
- In speech & audio coding, frequency bands might be used to simulate processing of human ear
- Reduce perceptually irrelevant content

Transform Encoder and Decoder



Linear Block Transforms

Linear Block Transform

- Each component of the N-dimensional output vector represents a linear combination of the N components of the N-dimensional input vector
- Can be written as matrix multiplication
- Analysis transform

$$\boldsymbol{u} = \boldsymbol{A} \cdot \boldsymbol{s} \tag{516}$$

- Synthesis transform

$$\boldsymbol{s'} = \boldsymbol{B} \cdot \boldsymbol{u'} \tag{517}$$

- Vector interpretation: $\boldsymbol{s'}$ is represented as a linear combination of column vectors of $\boldsymbol{B}$

$$\boldsymbol{s'} = \sum_{n=0}^{N-1} u'_n \cdot \boldsymbol{b}_n = u'_0 \cdot \boldsymbol{b}_0 + u'_1 \cdot \boldsymbol{b}_1 + \cdots + u'_{N-1} \cdot \boldsymbol{b}_{N-1} \tag{518}$$

Linear Block Transforms

Perfect Reconstruction Property

- Consider case that no quantization is applied ($\boldsymbol{u}' = \boldsymbol{u}$)
- Optimal synthesis transform:

$$\boldsymbol{B} = \boldsymbol{A}^{-1} \tag{519}$$

- Reconstructed samples are equal to source samples

$$\boldsymbol{s}' = \boldsymbol{B}\,\boldsymbol{u} = \boldsymbol{B}\,\boldsymbol{A}\,\boldsymbol{s} = \boldsymbol{A}^{-1}\,\boldsymbol{A}\,\boldsymbol{s} = \boldsymbol{s} \tag{520}$$

Optimal Synthesis Transform (in presence of quantization)

- Optimality: Minimum MSE distortion among all synthesis transforms
- $\boldsymbol{B} = \boldsymbol{A}^{-1}$ is optimal if
  - $\boldsymbol{A}$ is invertible and produces independent transform coefficients
  - the component quantizers are centroidal quantizers
- If above conditions are not fulfilled, a synthesis transform $\boldsymbol{B} \neq \boldsymbol{A}^{-1}$ may reduce the distortion

Orthogonal Block Transforms

Orthonormal Basis

- An analysis transform $\boldsymbol{A}$ forms an **orthonormal basis** if
  - basis vectors (matrix rows) are orthogonal to each other
  - basis vectors have to length 1
- The corresponding transform is called an **orthogonal transform**
- The transform matrices are called **unitary matrices**
- Unitary matrices with real entries are called **orthogonal matrix**
- Inverse of unitary matrices: Conjugate transpose

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^{\dagger} \quad \text{(for orthogonal matrices: } \boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}}\text{)} \tag{521}$$

Why are orthogonal transforms desirable?

- MSE distortion can be minimized by independent scalar quantization of the transform coefficients
- Orthogonality of the basis vectors sufficient: Vector norms can be taken into account in quantizer design

Properties of Orthogonal Block Transforms

- Transform coding with orthogonal transform and perfect reconstruction $\boldsymbol{B} = \boldsymbol{A}^{-1} = \boldsymbol{A}^{\dagger}$ preserves MSE distortion

$$\begin{aligned} d_N(\boldsymbol{s}, \boldsymbol{s'}) &= \frac{1}{N} \, (\boldsymbol{s} - \boldsymbol{s'})^{\dagger} \, (\boldsymbol{s} - \boldsymbol{s'}) \\ &= \frac{1}{N} \, \left(\boldsymbol{A}^{-1} \, \boldsymbol{u} - \boldsymbol{B} \, \boldsymbol{u'}\right)^{\dagger} \, \left(\boldsymbol{A}^{-1} \, \boldsymbol{u} - \boldsymbol{B} \, \boldsymbol{u'}\right) \\ &= \frac{1}{N} \, \left(\boldsymbol{A}^{\dagger} \, \boldsymbol{u} - \boldsymbol{A}^{\dagger} \, \boldsymbol{u'}\right)^{\dagger} \, \left(\boldsymbol{A}^{\dagger} \, \boldsymbol{u} - \boldsymbol{A}^{\dagger} \, \boldsymbol{u'}\right) \\ &= \frac{1}{N} \, (\boldsymbol{u} - \boldsymbol{u'})^{\dagger} \, \boldsymbol{A} \, \boldsymbol{A}^{-1} \, (\boldsymbol{u} - \boldsymbol{u'}) \\ &= \frac{1}{N} \, (\boldsymbol{u} - \boldsymbol{u'})^{\dagger} \, (\boldsymbol{u} - \boldsymbol{u'}) \\ &= d_N(\boldsymbol{u}, \boldsymbol{u'}) \end{aligned} \tag{522}$$

- Scalar quantization that minimizes MSE in transform domain also minimizes MSE in original signal space
- For the special case of orthogonal matrices: $(\cdots)^{\dagger} = (\cdots)^{\mathrm{T}}$

Properties of Orthogonal Block Transforms

- Covariance matrix of transform coefficients

$$\begin{aligned} \boldsymbol{C}_{UU} &= E\big\{(\boldsymbol{U}-E\{\boldsymbol{U}\})(\boldsymbol{U}-E\{\boldsymbol{U}\})^{\mathrm{T}}\big\} \\ &= E\Big\{\boldsymbol{A}\,(\boldsymbol{S}-E\{\boldsymbol{S}\})(\boldsymbol{S}-E\{\boldsymbol{S}\})^{\mathrm{T}}\,\boldsymbol{A}^{\mathrm{T}}\Big\} \\ &= \boldsymbol{A}\,\boldsymbol{C}_{SS}\,\boldsymbol{A}^{-1} \end{aligned} \tag{523}$$

- Since the trace of a matrix is similarity-invariant,

$$\mathrm{tr}(\boldsymbol{X}) = \mathrm{tr}(\boldsymbol{P}\,\boldsymbol{X}\,\boldsymbol{P}^{-1}), \tag{524}$$

  and the trace of an autocovariance matrix is the sum of the variances of the vector components, we have

$$\frac{1}{N}\sum_{i=0}^{N-1}\sigma_i^2 = \sigma_S^2. \tag{525}$$

- **The arithmetic mean of the variances of the transform coefficients is equal to the variances of the source**

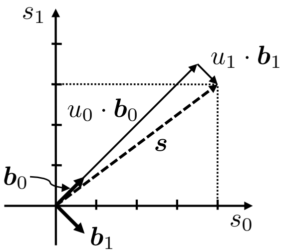
Geometrical Interpretation of Orthogonal Transforms

- Inverse 2-d transform matrix (= transpose of forward transform matrix)

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_0 & \boldsymbol{b}_1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \boldsymbol{A}^{\mathrm{T}}$$

- Vector interpretation for 2-d example

$$\begin{aligned} \boldsymbol{s} &= u_0 \cdot \boldsymbol{b}_0 + u_1 \cdot \boldsymbol{b}_1 \\ \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} &= u_0 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + u_1 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 4 \\ 3 \end{bmatrix} &= 3.5 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.5 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$



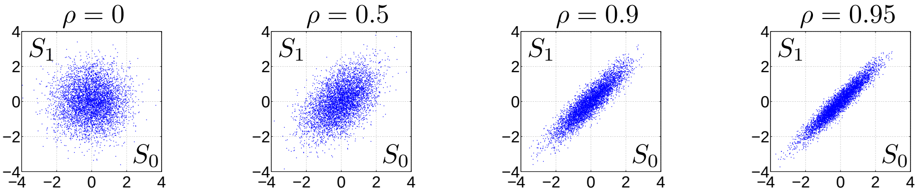
yielding transform coefficients

$$u_0 = \sqrt{2} \cdot 3.5 \qquad u_1 = \sqrt{2} \cdot 0.5$$

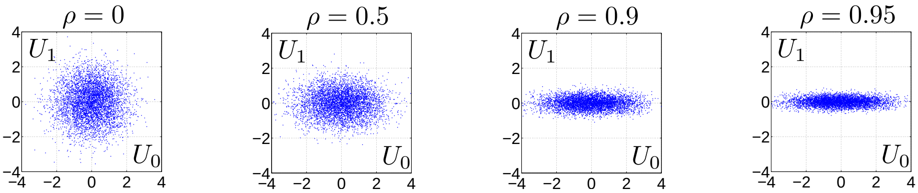
- An orthogonal transform is a rotation from the signal coordinate system into the coordinate system of the basis functions

Transform Example for $N = 2$

Adjacent samples of Gauss-Markov source with different correlation factors ρ

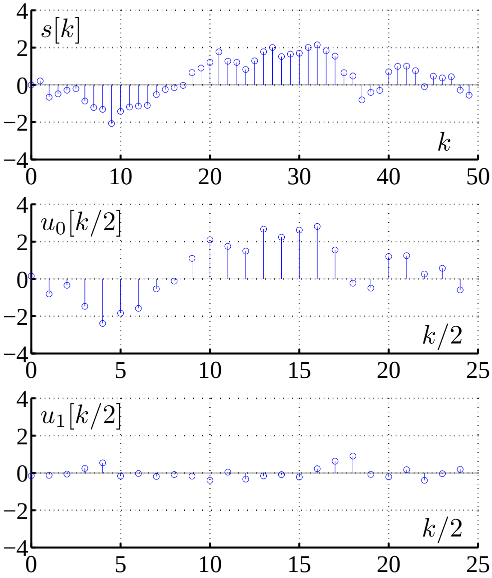


Transform coefficients for orthonormal 2D transform



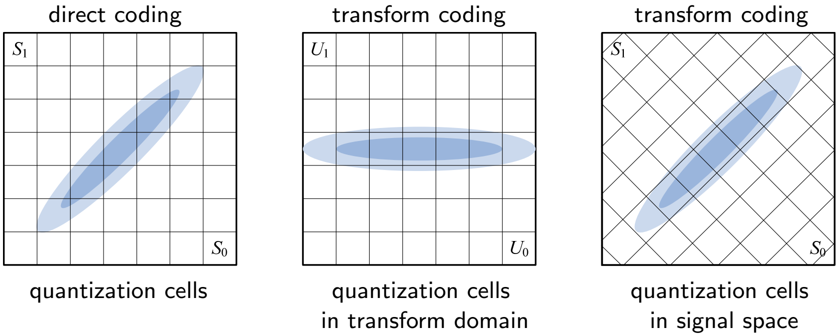
Example for Waveforms (Gauss-Markov Source with $\rho = 0.95$)

- Top: signal $s[k]$
- Middle:
  transform coefficient $u_0[k/2]$
  also called dc coefficient
- Bottom:
  transform coefficient $u_1[k/2]$
  also called ac coefficient
- Number of transform coefficients u_0 is half the number of samples s
- Number of transform coefficients u_1 is half the number of samples s



Scalar Quantization in Transform Domain

Consider Transform Coding with Orthogonal Transforms



- Quantization cells are
  - hyper-rectangles as in scalar quantization
  - but rotated and aligned with the transform basis vectors
- Number of quantization cells with appreciable probabilities is reduced
  $\Longrightarrow$ indicates improved coding efficiency for correlated sources

Bit Allocation for Transform Coefficients

- Problem: Distribute bit rate R among the N transform coefficients such that the resulting distortion D is minimized

$$\min \quad D(R) = \frac{1}{N}\sum_{i=1}^{N} D_i(R_i) \quad \text{subject to} \quad \frac{1}{N}\sum_{i=1}^{N} R_i \leq R \tag{526}$$

with $D_i(R_i)$ being the oper. distortion-rate functions of the scalar quantizers

- Approach: Minimize Lagrangian cost function: $J = D + \lambda R$

$$\frac{\partial}{\partial R_i}\left(\sum_{i=1}^{N} D_i(R_i) + \lambda \sum_{i=1}^{N} R_i\right) = \frac{\partial D_i(R_i)}{\partial R_i} + \lambda \stackrel{!}{=} 0 \tag{527}$$

- Solution: Pareto condition

$$\frac{\partial D_i(R_i)}{\partial R_i} = -\lambda = \text{const} \tag{528}$$

- Move bits from coefficients with small distortion reduction per bit to coefficients with larger distortion reduction per bit

Bit Allocation for Transform Coefficients

- Operational distortion-rate function of scalar quantizers can be written as

$$D_i(R_i) = \sigma_i^2 \cdot g_i(R_i) \qquad (529)$$

- Justified to assume that $g_i(R_i)$
  - is a continuous strictly convex function and
  - has a continuous strictly increasing derivative $g_i'(R_i)$ with $g_i'(\infty) = 0$
- Pareto condition becomes

$$-\sigma_i^2 \cdot g_i'(R_i) = \lambda \qquad (530)$$

- If $\lambda \geq -\sigma_i^2\, g_i'(0)$, the quantizer for u_i cannot be operated at the given slope $\Longrightarrow$ Set the corresponding component rate to $R_i = 0$
- Bit allocation rule

$$R_i = \begin{cases} 0 & : \quad -\sigma_i^2 g_i'(0) \leq \lambda \\ \eta_i\left(-\frac{\lambda}{\sigma_i^2}\right) & : \quad -\sigma_i^2 g_i'(0) > \lambda \end{cases} \qquad (531)$$

  where $\eta_i(\cdot)$ denotes the inverse of the derivative $g_i'(\cdot)$
- Similar to reverse water-filling for Gaussian random variables

Approximation for Gaussian Sources

- Transform coefficients have also a Gaussian distribution
- Experimentally found approximation for entropy-constrained scalar quantization for Gaussian sources ($a \approx 0.952$)

$$g(R) = \frac{\pi e}{6a} \ln(a \cdot 2^{-2R} + 1) \tag{532}$$

- Use parameter

$$\theta = \lambda \, \frac{3\,(a+1)}{\pi e \ln 2} \qquad \text{with} \qquad 0 \leq \theta \leq \sigma^2_{\max} \tag{533}$$

- Bit allocation rule

$$R_i(\theta) = \begin{cases} 0 & : \quad \theta \geq \sigma_i^2 \\ \frac{1}{2}\log_2\left(\frac{\sigma_i^2}{\theta}(a+1) - a\right) & : \quad \theta < \sigma_i^2 \end{cases} \tag{534}$$

- Resulting component distortions

$$D_i(\theta) = \begin{cases} \sigma_i^2 & : \quad \theta \geq \sigma_i^2 \\ -\frac{\varepsilon^2 \ln 2}{a} \cdot \sigma_i^2 \cdot \log_2\left(1 - \frac{\theta}{\sigma_i^2}\frac{a}{a+1}\right) & : \quad \theta < \sigma_i^2 \end{cases} \tag{535}$$

High-Rate Approximation

- Assumption: High-rate approximation valid for all component quantizers
- High-rate approximation for distortion-rate function of component quantizers

$$D_i(R_i) = \varepsilon_i^2 \cdot \sigma_i^2 \cdot 2^{-2R_i} \qquad (536)$$

  where ε_i^2 depends on transform coefficient distribution and quantizer
- Pareto condition

$$\frac{\partial}{\partial R_i} D_i(R_i) = -2\,\ln 2\,\varepsilon_i^2\,\sigma_i^2\,2^{-2R_i} = -2\,\ln 2\,D_i(R_i) = -\lambda = \text{const} \qquad (537)$$

  states that all quantizers are operated at the same distortion
- Bit allocation rule

$$R_i(D) = \frac{1}{2}\,\log_2\left(\frac{\varepsilon_i^2\,\sigma_i^2}{D}\right) \qquad (538)$$

- Overall operational rate-distorion function

$$R(D) = \frac{1}{N}\sum_{i=0}^{N-1} R_i(D) = \frac{1}{2N}\sum_{i=0}^{N-1}\log_2\left(\frac{\sigma_i^2\varepsilon_i^2}{D}\right) \qquad (539)$$

High-Rate Approximation

- Overall operational rate-distorion function

$$R(D) = \frac{1}{2N} \sum_{i=0}^{N-1} \log_2 \left(\frac{\sigma_i^2 \varepsilon_i^2}{D} \right) = \frac{1}{2} \log_2 \left(\frac{\tilde{\varepsilon}^2 \, \tilde{\sigma}^2}{D} \right) \tag{540}$$

with geometric means

$$\tilde{\sigma}^2 = \left(\prod_{i=0}^{N-1} \sigma_i^2 \right)^{\frac{1}{N}} \qquad \text{and} \qquad \tilde{\varepsilon}^2 = \left(\prod_{i=0}^{N-1} \varepsilon_i^2 \right)^{\frac{1}{N}} \tag{541}$$

- Overall distortion-rate function

$$D(R) = \tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2 \cdot 2^{-2R} \tag{542}$$

- For Gaussian sources (transform coefficients are also Gaussian) and entropy-constrained scalar quantizers, we have $\varepsilon_i^2 = \varepsilon^2 = \frac{\pi e}{6}$, yielding

$$D_G(R) = \frac{\pi e}{6} \cdot \tilde{\sigma}^2 \cdot 2^{-2R} \tag{543}$$

Transform Coding Gain at High Rates

- Transform coding gain is the ratio of the distortion for scalar quantization and the distortion for transform coding

$$G_T = \frac{\varepsilon_S^2 \cdot \sigma_S^2 \cdot 2^{-2R}}{\tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2 \cdot 2^{-2R}} = \frac{\varepsilon_S^2 \cdot \sigma_S^2}{\tilde{\varepsilon}^2 \cdot \tilde{\sigma}^2} \quad (544)$$

with

σ_S^2 : variance of the input signal

ε_S^2 : factor of high-rate approximation for direct scalar quantization

- High-rate transform coding gain for Gaussian sources

$$G_T = \frac{\sigma_S^2}{\tilde{\sigma}^2} = \frac{\frac{1}{N}\sum_{i=0}^{N-1}\sigma_i^2}{\sqrt[N]{\prod_{i=0}^{N-1}\sigma_i^2}} \quad (545)$$

Ratio of arithmetic and geometric mean of the transform coefficient variances

- The high-rate transform coding gain for Gaussian sources is maximized if the geometric mean is minimized ($\Longrightarrow$ Karhunen Loève Transform)

Example: Orthogonal Transform with $N = 2$

- Input vector and transform matrix

$$\boldsymbol{s} = \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{546}$$

- Transformation

$$\boldsymbol{u} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \boldsymbol{A} \cdot \boldsymbol{s} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \end{bmatrix} \tag{547}$$

- Coefficients

$$u_0 = \frac{1}{\sqrt{2}}(s_0 + s_1), \qquad u_0 = \frac{1}{\sqrt{2}}(s_0 - s_1) \tag{548}$$

- Inverse transformation

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{549}$$

Example: Orthogonal Transform with $N = 2$

- Variance of transform coefficients

$$\sigma_0^2 = E\{U_0^2\} = E\left\{\frac{1}{2}(S_0+S_1)^2\right\} = \frac{1}{2}\left(E\{S_0^2\} + E\{S_1^2\} + 2E\{S_0 S_1\}\right)$$
$$= \frac{1}{2}\left(\sigma_S^2 + \sigma_S^2 + 2\sigma_S^2\rho\right) = \sigma_S^2(1+\rho) \qquad (550)$$
$$\sigma_1^2 = E\{U_1^2\} = \sigma_S^2(1-\rho) \qquad (551)$$

- Cross-correlation of transform coefficients

$$E\{U_0 U_1\} = \frac{1}{2}E\{(S_0+S_1)\cdot(S_0-S_1)\}$$
$$= \frac{1}{2}E\left\{\left(S_0^2 - S_1^2\right)\right\} = \sigma_S^2 - \sigma_S^2 = 0 \qquad (552)$$

- Transform coding gain for Gaussian (assuming optimal bit allocation)

$$G_T = \frac{\sigma_S^2}{\sqrt{\sigma_0^2 + \sigma_1^2}} = \frac{1}{\sqrt{1-\rho^2}} \qquad (553)$$

Example: Analysis of Transform Coding for $N = 2$

- Rate-distortion cost before transform

$$J^{(0)} = 2(D + \lambda R) \quad \text{(for 2 samples)}$$

- Rate-distortion cost after transform

$$J^{(1)} = (D_0 + D_1) + \lambda(R_0 + R_1) \qquad \text{(for both transform coefficients)}$$

- Gain in r-d cost due to transform at same rate $(R_0 + R_1 = R)$

$$\Delta J = J^{(0)} - J^{(1)} = 2D - D_0 - D_1 \tag{554}$$

- For Gaussian sources, input and output of transform have Gaussian pdf
- With operational distortion-rate function for an entropy-constrained scalar quantizer at high rates ($D = \varepsilon^2 \cdot \sigma^2 \cdot 2^{-2R}$ with $\varepsilon^2 = \pi e/6$), we have

$$\Delta J = \varepsilon^2 \sigma_S^2 \big(2^{-2R+1} - (1+\rho)2^{-2R_0} - (1-\rho)2^{-2R_1}\big) \tag{555}$$

- By eliminating R_1 using $R_1 = 2R - R_0$, we get

$$\Delta J \;=\; \varepsilon^2 \sigma_S^2 \big(2^{-2R+1} - (1+\rho)2^{-2R_0} - (1-\rho)2^{-2(2R-R_0)}\big) \tag{556}$$

Example: Analysis of Transform Coding for $N = 2$

- Gain in rate-distortion cost due to transform

$$\Delta J \;=\; \varepsilon^2 \sigma_S^2 \big(2^{-2R+1} - (1+\rho) 2^{-2R_0} - (1-\rho) 2^{-2(2R-R_0)} \big) \tag{557}$$

- To maximize gain, we set

$$\frac{\partial}{\partial R_0} \Delta J = 2 \ln 2 \cdot (1+\rho) 2^{-2R_0} - 2 \ln 2 \cdot (1-\rho) 2^{-4R+2R_0} \overset{!}{=} 0 \tag{558}$$

  yielding the bit allocation rule

$$R_0 = R + \frac{1}{2} \log_2 \sqrt{\frac{1+\rho}{1-\rho}} \tag{559}$$

- Same expression is obtained by using the previously derived high rate bit allocation rule

$$R_i = \frac{1}{2} \log_2 \left(\frac{\varepsilon^2 \, \sigma_i^2}{D} \right) \tag{560}$$

- Operational high-rate distortion-rate function (Gaussian, ECSQ, $N = 2$)

$$D(R) = \frac{\pi e}{6} \cdot \sqrt{1-\rho^2} \cdot \sigma_S^2 \cdot 2^{-2R} \tag{561}$$

General Bit Allocation for Transform Coefficients

For Gaussian sources, the following points need to be considered:

- High-rate approximations are not valid for low bit rates; better approximations should be used for low rates
- For low rates, Pareto conditions cannot be fulfilled for all transform coefficients, since the component rates R_i must not be less then 0
- Solution:
  - Use generalized approximation of $D_i(R_i)$ for components quantizers
  - Set components rates R_i to zero for all transform coefficients, for which the Pareto condition $\frac{\partial}{\partial R_i} D(R_i) = -\lambda$ cannot be fullfilled for $R_i \geq 0$
  - Distribute rate among remaining coefficients

For non-Gaussian sources, the following needs to be considered in addition

- The transform coefficients have different (non-Gaussian) distributions (except for large transform sizes)
- Using the same quantizer design for all transform coefficients with $D_i(R_i) = \sigma_i^2 g(R_i)$ is suboptimal

Karhunen Loève Transform (KLT)

- Karhunen Loève Transform
  - Orthogonal transform that decorrelates the input vectors
  - Transform matrix depends on the source
- Autocorrelation matrix of input vectors $\boldsymbol{s}$

$$\boldsymbol{R}_{SS} = E\left\{\boldsymbol{S}\boldsymbol{S}^{\mathrm{T}}\right\} \tag{562}$$

- Autocorrelation matrix of transform coefficient vectors $\boldsymbol{u}$

$$\begin{aligned} \boldsymbol{R}_{UU} &= E\left\{\boldsymbol{U}\boldsymbol{U}^{\mathrm{T}}\right\} = E\left\{(\boldsymbol{A}\boldsymbol{S})(\boldsymbol{A}\boldsymbol{S})^{\mathrm{T}}\right\} = \boldsymbol{A}\cdot E\left\{\boldsymbol{S}\boldsymbol{S}^{\mathrm{T}}\right\}\cdot\boldsymbol{A}^{\mathrm{T}} \\ &= \boldsymbol{A}\,\boldsymbol{R}_{SS}\,\boldsymbol{A}^{\mathrm{T}} \end{aligned} \tag{563}$$

- By multiplying with $\boldsymbol{A}^{-1} = \boldsymbol{A}^{\mathrm{T}}$ from the front, we get

$$\boldsymbol{R}_{SS}\cdot\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}}\cdot\boldsymbol{R}_{UU} \tag{564}$$

- To get uncorrelated transform coefficients, we need to obtain a diagonal autocorrelation matrix $\boldsymbol{R}_{UU}$ for the transform coefficients

Karhunen Loève Transform (KLT)

- Expression for autocorrelation matrices

$$\boldsymbol{R}_{SS} \cdot \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{R}_{UU} \tag{565}$$

- $\boldsymbol{R}_{UU}$ is a diagonal matrix if the eigenvector equation

$$\boldsymbol{R}_{SS} \cdot \boldsymbol{b}_i = \xi_i \cdot \boldsymbol{b}_i \tag{566}$$

  is fulfilled for all basis vectors $\boldsymbol{b}_i$ (column vectors of $\boldsymbol{A}^{\mathrm{T}}$, row vectors of $\boldsymbol{A}$)

- The transform matrix $\boldsymbol{A}$ decorrelates the input vectors if its rows are equal to the unit-norm eigenvectors $\boldsymbol{v}_i$ of $\boldsymbol{R}_{SS}$

$$\boldsymbol{A}_{KLT} = \Big[\boldsymbol{v}_0 \; \boldsymbol{v}_1 \; \cdots \; \boldsymbol{v}_{N-1} \Big]^{\mathrm{T}} \tag{567}$$

- The resulting autocorrelation matrix $\boldsymbol{R}_{UU}$ is a diagonal matrix with the eigenvalues of $\boldsymbol{R}_{SS}$ on its main diagonal

$$\boldsymbol{R}_{UU} = \begin{bmatrix} \xi_0 & 0 & \cdots & 0 \\ 0 & \xi_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_{N-1} \end{bmatrix} \tag{568}$$

Optimality of KLT for Gaussian Sources

- Transform coding with orthogonal $N \times N$ transform matrix $\boldsymbol{A}$ and $\boldsymbol{B} = \boldsymbol{A}^T$
- Scalar quantization using scaled quantizers

$$D(R, \boldsymbol{A}_k) = \sum_{i=0}^{N-1} \sigma_i^2(\boldsymbol{A}_k) \cdot g(R_i) \tag{569}$$

with $\sigma_i^2(\boldsymbol{A}_k)$ being variance of i-th transform coefficient and $\boldsymbol{A}_k$ being the transform matrix

- Consider an arbitrary orthogonal transform matrix $\boldsymbol{A}_0$ and an arbitrary bit allocation given by the vector $\boldsymbol{r} = [R_0, \cdots, R_{N-1}]^{\mathrm{T}}$ with $\sum_{i=0}^{N-1} R_i = R$
- Starting with arbitrary orthogonal matrix $\boldsymbol{A}_0$, apply iterative algorithm that generates a series of orthonormal transform matrices $\{\boldsymbol{A}_k\},\ k = 1, 2, ...$
- Iteration $\boldsymbol{A}_{k+1} = \boldsymbol{J}_k \boldsymbol{A}_k$ consists of Jacobi rotation and re-ordering
$\Longrightarrow$ Transform matrix approaches a KLT matrix
- Can show that for all $\boldsymbol{A}_k$: $D(R, \boldsymbol{A}_{k+1}) \leq D(R, \boldsymbol{A}_{k+1})$
$\Longrightarrow$ KLT is optimal transform for Gaussian sources (minimizes MSE)

Asymp. High-Rate Performance of KLT for Gaussian Sources

- Transform coefficient variances σ_i^2 are equal to the eigenvalues ξ_i of $\boldsymbol{R}_{SS}$
- High-rate approximation for Gaussian source and optimal ECSQ

$$\begin{aligned} D(R) &= \frac{\pi e}{6} \cdot \tilde{\sigma}^2 \cdot 2^{-2R} = \frac{\pi e}{6} \cdot \tilde{\xi} \cdot 2^{-2R} \\ &= \frac{\pi e}{6} \cdot 2^{\frac{1}{N}\sum_{i=0}^{N-1} \log_2 \xi_i} \cdot 2^{-2R} \end{aligned} \tag{570}$$

- For $N \to \infty$, we can apply the theorem of Szegö and Grenander for infinite Toeplitz matrices: If all eigenvalues ξ_i of an infinite autocorrelation matrix are finite and $G(\xi_i)$ is any continuous function over all eigenvalues,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} G(\xi_i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\Phi(\omega)) d\omega \tag{571}$$

- Resulting distortion-rate function for KLT of infinite size for high rates

$$D_{\mathrm{KLT}}^{\infty}(R) = \frac{\pi e}{6} \cdot 2^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log_2 \Phi_{SS}(\omega) \cdot \mathrm{d}\omega} \cdot 2^{-2R} \tag{572}$$

Asymp. High-Rate Performance of KLT for Gaussian Sources

- Asymptotic distortion-rate function for KLT of infinite size for high rates

$$D_{\mathrm{KLT}}^{\infty}(R) = \frac{\pi e}{6} \cdot 2^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log_2 \Phi_{SS}(\omega)\cdot \mathrm{d}\omega} \cdot 2^{-2R} \tag{573}$$

- Information distortion-rate function (fundamental bound) is by a factor $\varepsilon^2 = \pi e/6$ smaller

$$D(R) = 2^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log_2 \Phi_{SS}(\omega)\cdot \mathrm{d}\omega} \cdot 2^{-2R} \tag{574}$$

- Asymptotic transform gain ($N \to \infty$) at high rates

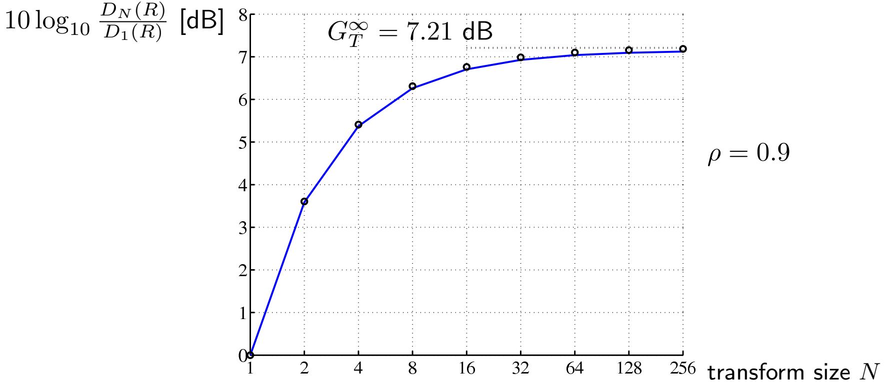
$$G_T^{\infty} = \frac{\varepsilon^2 \sigma_S^2 2^{-2R}}{D_{\mathrm{KLT}}^{\infty}(R)} = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{SS}(\omega) d\omega}{2^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log_2 \Phi_{SS}(\omega) d\omega}} \tag{575}$$

- Asymptotic transform gain ($N \to \infty$) at high rates is identical to the asymptotic prediction gain at high rates

High-Rate KLT Transform Gain for Gauss-Markov Sources

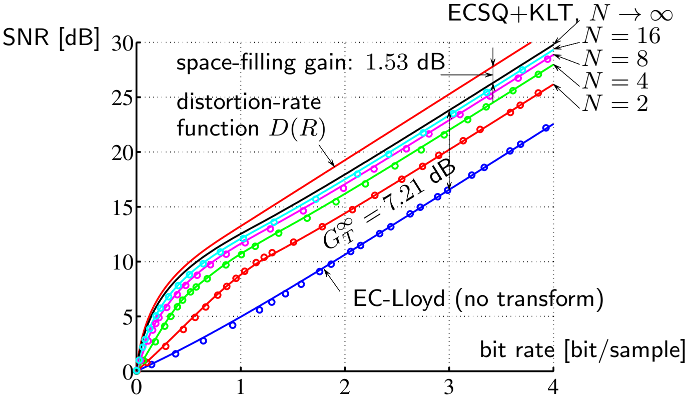
- Operational distortion-rate function for KLT of size N, ECSQ, and optimum bit allocation for Gauss-Markov sources with correlation factor ρ

$$D_N(R) = \frac{\pi e}{6} \cdot \sigma_S^2 \cdot (1-\rho^2)^{1-1/N} \cdot 2^{-2R} \tag{576}$$

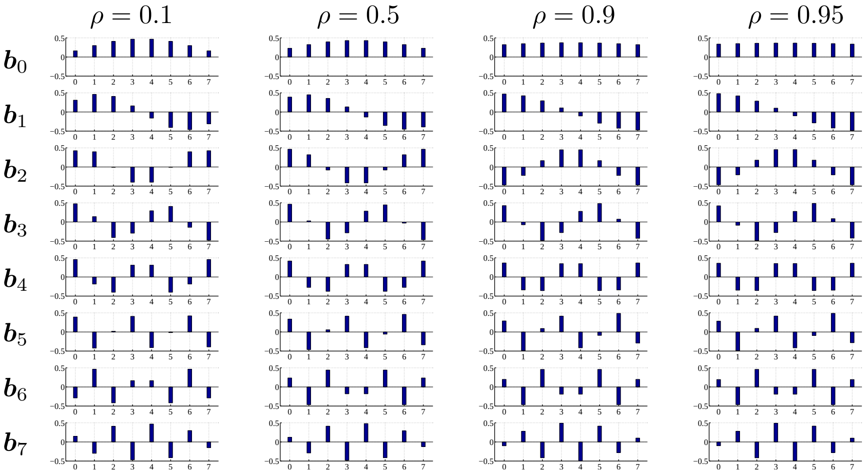


Operat. Distortion-Rate Functions for Gauss-Markov

- Distortion-rate curves for coding a first-order Gauss-Markov source with correlation factor $\rho = 0.9$ and different transform sizes N



KLT Basis Functions for Gauss-Markov Sources and Size $N = 8$



Walsh-Hadamard Transform

- Very simple orthogonal transform (only additions & final scaling)
- For transform sizes N that are positive integer power of 2

$$\boldsymbol{A}_N = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{A}_{N/2} & \boldsymbol{A}_{N/2} \\ \boldsymbol{A}_{N/2} & -\boldsymbol{A}_{N/2} \end{bmatrix} \qquad \text{with} \qquad \boldsymbol{A}_1 = [1]. \tag{577}$$

- Transform matrix for $N = 8$

$$\boldsymbol{A}_8 = \frac{1}{2\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \tag{578}$$

- Piecewise-constant basis vectors
- Image & video coding: Produces subjectively disturbing artifacts when combined with strong quantization

Discrete Fourier Transform (DFT)

- Discrete version of the Fourier transform
- Forward Transform

$$u[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi kn}{N}} \tag{579}$$

- Inverse Transform

$$s[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} u[k] \cdot e^{j\frac{2\pi kn}{N}} \tag{580}$$

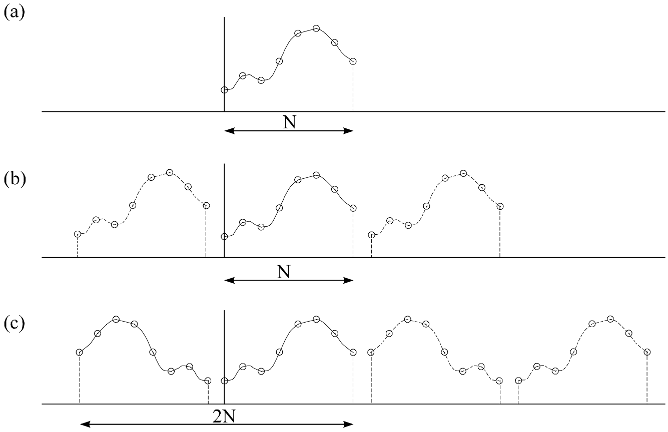
- DFT is an orthonormal transform (specified by a unitary transform matrix)
- Produces complex transform coefficients
- For real inputs, it obeys the symmetry $u[k] = u^*[N-k]$, so that N real samples are mapped onto N real values
- FFT is a fast algorithm for DFT computation, uses sparse matrix factorization
- Implies periodic signal extension: Differences between left and right signal boundary reduces rate of convergence of Fourier series
- Strong quantization $\Longrightarrow$ Significant high-frequent artifacts

Discrete Fourier Transform vs. Discrete Cosine Transform

(a) Input time-domain signal

(b) Time-domain replica in case of DFT

(c) Time-domain replica in case of DCT-II



Derivation of DCT Type II

- Reduce quantization errors of DFT by introducing mirror symmetry and applying a DFT of approximately double size
- Signal with mirror symmetry

$$s^*[n] = \begin{cases} s[n-1/2] & : \quad 0 \le n < N \\ s[2N-n-3/2] & : \quad N \le n < 2N \end{cases} \tag{581}$$

- Transform coefficients (orthonormal: divide $u^*[0]$ by $\sqrt{2}$)

$$\begin{aligned} u^*[k] &= \frac{1}{\sqrt{2N}} \sum_{i=0}^{2N-1} s^*[i] e^{-j\frac{2\pi k n}{2N}} \\ &= \frac{1}{\sqrt{2N}} \sum_{n=0}^{N-1} s[n-1/2] \left(e^{-j\frac{\pi}{N}kn} + e^{-j\frac{\pi}{N}k(2N-n-1)} \right) \\ &= \frac{1}{\sqrt{2N}} \sum_{n=0}^{N-1} s[n] \left(e^{-j\frac{\pi}{N}k\left(n+\frac{1}{2}\right)} + e^{j\frac{\pi}{N}k\left(n+\frac{1}{2}\right)} \right) \\ &= \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} s[n] \cos\left(\frac{\pi}{N} k \left(n + \frac{1}{2} \right) \right) \end{aligned} \tag{582}$$

Discrete Cosine Transform (DCT)

- Implicit periodicity of DFT leads to loss in coding efficiency
- This can be reduced by introducing mirror symmetry at the boundaries and applying a DFT of approximately double size
- Due to mirror symmetry, imaginary sine terms get eliminated and only cosine terms remain
- Most common DCT is the so-called DCT-II (mirror symmetry with sample repetitions at both sides: $n = -\frac{1}{2}$)
- DCT and IDCT Type-II are given by

$$u[k] = \alpha_k \sum_{n=0}^{N-1} s[n] \cdot \cos\left[k \cdot \left(n + \frac{1}{2}\right) \cdot \frac{\pi}{N}\right] \quad (583)$$

$$s[n] = \sum_{k=0}^{N-1} \alpha_k \cdot u[k] \cdot \cos\left[k \cdot \left(n + \frac{1}{2}\right) \cdot \frac{\pi}{N}\right] \quad (584)$$

where $\alpha_0 = \sqrt{\frac{1}{N}}$ and $\alpha_n = \sqrt{\frac{2}{N}}$ for $n \neq 0$

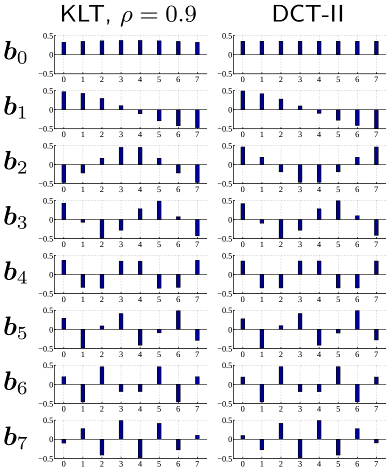
Comparison of DCT and KLT

- Correlation matrix of a first-order Markov processes can be written as

$$\boldsymbol{R}_{SS} = \sigma_S^2 \cdot \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{N-1} \\ \rho & 1 & \rho & \cdots & \rho^{N-2} \\ \vdots & & & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \cdots & 1 \end{bmatrix} \tag{585}$$

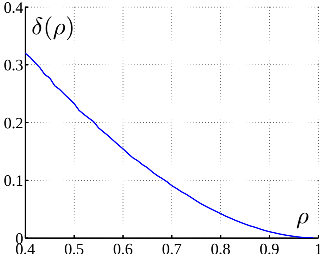
- DCT is a good approximation of the eigenvectors of $\boldsymbol{R}_{SS}$
- DCT basis vectors approach the basis functions of the KLT for first-order Markov processes with $\rho \rightarrow 1$
- DCT does not depend on input signal
- Fast algorithms for computing forward and inverse transform
- Justification for wide usage of DCT (or integer approximations thereof) in image and video coding:
  JPEG, H.261, H.262/MPEG-2, H.263, MPEG-4, H.264/AVC, H.265/HEVC

KLT Convergence Towards DCT for $\rho \to 1$



Difference between the transform matrices of KLT and DCT-II

$$\delta(\rho) = ||\boldsymbol{A}_{KLT}(\rho) - \boldsymbol{A}_{DCT}||_2^2$$



Two-dimensional Transforms

- 2-D linear transform:
  Input image is represented as a linear combination of basis images
- An orthonormal transform is separable and symmetric, if the transform of a signal block $\boldsymbol{s}$ of size $N \times N$ can be expressed as,

$$\boldsymbol{u} = \boldsymbol{A} \cdot \boldsymbol{s} \cdot \boldsymbol{A}^{\mathrm{T}} \tag{586}$$

  where $\boldsymbol{A}$ is the transformation matrix and $\boldsymbol{u}$ is the matrix of transform coefficients, both of size $N \times N$.
- The inverse transform is

$$\boldsymbol{s} = \boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{s} \cdot \boldsymbol{A} \tag{587}$$

- Great practical importance:
  Transform requires 2 matrix multiplications of size $N \times N$ instead one multiplication of a vector of size $1 \times N^2$ with a matrix of size $N^2 \times N^2$
- Reduction of the complexity from $O(N^4)$ to $O(N^3)$

2-dimensional DCT Example

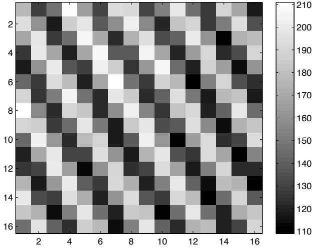
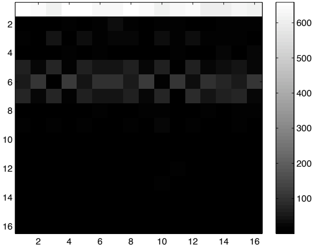


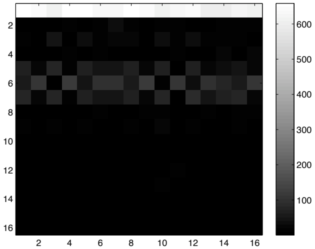
image block



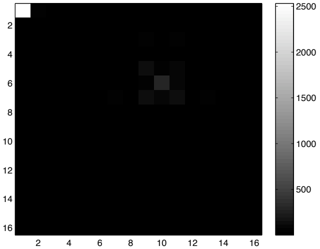
column-wise DCT

- 1-d DCT is applied to each column of an image block
- Notice the energy concentration in the first row (DC coefficients)

2-dimensional DCT Example



column-wise DCT



final result

- For convenience, column-wise DCT result is repeated on left side
- 1-d DCT is applied to each row of the intermediate result
- Notice the energy concentration in the first coefficient

Entropy Coding of Transform Coefficients

- AC coefficients are very likely equal to zero (for moderate quantization)
- For 2-d, ordering of the transform coefficients by zig-zag (or similar) scan
- Example for zig-zag scanning in case of a 2-d transform

185	3	1	1	−3	2	−1	0
1	1	−1	0	−1	0	0	1
0	0	1	0	−1	0	0	0
1	1	0	−1	0	0	0	−1
0	0	1	0	0	0	−1	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

- Huffman code for events {number of leading zeros, coefficient value} or events {end-of-block, number of leading zeros, coefficient value}
- Arithmetic coding: For example, use probabilities that particular coefficient is unequal to zero when quantizing with a particular step size

Chapter Summary

Orthogonal block transform

- Orthogonal transform: Rotation of coordinate system in signal space
- Purpose of transform: Decorrelation, energy concentration
  $\Longrightarrow$ Align quantization cells with primary axis of joint pdf
- KLT achieves optimum decorrelation, but is signal dependent
- DCT shows reduced blocking artifacts compared to DFT
- For Gauss-Markov and $\rho \rightarrow \infty$: DCT approaches KLT

Bit allocation and transform coding gain

- For Gaussian sources: Bit allocation proportional to logarithm of variances
- For high rates: Optimum bit allocation yields equal component distortion
- Larger transform size increases gain for Gauss-Markov source

Application of transform coding

- Widely used in image and video coding:
  DCT (or approximation) + quantization + (zig-zag) scan + entropy coding
  $\Longrightarrow$ JPEG, H.262/MPEG-2, H.263, MPEG-4, H.264/AVC, H.265/HEVC

Exercise 25

Consider a zero-mean Gauss-Markov process with variance σ_S^2 and correlation coefficient ρ. The source is coded using a transform coding system consisting of a N-dimensional KLT, optimal bit allocation and optimal entropy-constrained scalar quantizers with optimal entropy coding.

Show that the high-rate approximation of the operational distortion-rate function is given by

$$D(R) = \frac{\pi\, e}{6} \cdot \sigma_S^2 \cdot \left(1 - \rho^2\right)^{\frac{N-1}{N}} \cdot 2^{-2R}$$

Exercise 26

In the video coding standard ITU-T Rec. H.264 the following forward transform is used (more accurately, only the inverse transform is specified in the standard, but the given transform is used in most actual encoder implementation),

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ 1 & -1 & -1 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix}$$

How large is the high-rate transform coding gain (in dB) for a zero-mean Gauss-Markov process with the correlation factor $\rho = 0.9$?

By what amount (in dB) can the high-rate transform coding gain be increased if the transform is replaced by a KLT?

NOTE: The basis functions of the given transform are orthogonal to each other, but they don't have the same norm.

Exercise 27 – Part 1/2

Given is a zero-mean Gaussian process with the autocovariance matrix for $N = 4$

$$\mathbf{C}_{SS} = \sigma_S^2 \begin{bmatrix} 1.00 & 0.95 & 0.92 & 0.88 \\ 0.95 & 1.00 & 0.95 & 0.92 \\ 0.92 & 0.95 & 1.00 & 0.95 \\ 0.88 & 0.92 & 0.95 & 1.00 \end{bmatrix}$$

Consider transform coding with the Hadamard transform given by

$$\mathbf{A} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

The scalar quantizers for the transform coeff. have 5 operation points given by

$$\begin{aligned} R_i = 0 \quad &\Longrightarrow \quad D_i = \sigma_i^2 \\ R_i = 1 \quad &\Longrightarrow \quad D_i = 0.32\,\sigma_i^2 \\ R_i = 2 \quad &\Longrightarrow \quad D_i = 0.09\,\sigma_i^2 \\ R_i = 3 \quad &\Longrightarrow \quad D_i = 0.02\,\sigma_i^2 \\ R_i = 4 \quad &\Longrightarrow \quad D_i = 0.01\,\sigma_i^2 \end{aligned}$$

Exercise 27 – Part 2/2

For each transform coefficients, any of the 5 operation points can be chosen.

Derive the optimal bit allocation (i.e., the component rates R_i for $i = 0, 1, 2, 3$) for the overall rate of $R = 1$ bit per sample.

What distortion D and SNR is achieved for this rate?

How big is the transform coding gain? Is it larger than, smaller than, or equal to the transform coding gain for high rates (the above given operation points are good approximations for optimal entropy-constrained quantizers for Gaussian sources and can be considered as valid for the comparison)?

Exercise 28

Consider transform coding with an orthogonal transform of a zero-mean Gaussian source with variance σ_S^2. The used scalar quantizers have the operational distortion rate function

$$D_i(R_i) = \sigma_i^2 \, g(R_i)$$

where $g(R)$ is some not further specified function.

We don't use an optimal bit allocation, but assign the same rate to all transform coefficients.

Does the transform coding still provide a gain in comparison to simple scalar quantization with the given quantizer, assuming that the Gaussian source is not iid?

Exercise 29

Consider a zero-mean Gauss-Markov process with variance $\sigma_S^2 = 1$ and correlation coefficient $\rho = 0.9$. As transform a KLT of size 3 is used, the resulting transform coefficient variances are

$$\sigma_0^2 = 2.7407, \qquad \sigma_1^2 = 0.1900, \qquad \sigma_2^2 = 0.0693$$

Consider high-rate quantization with optimal entropy-constrained scalar quantizers.

Derive the high-rate operational distortion rate function. What is the optimal high-rate bit allocation scheme for a given overall rate R?

Determine the component rates, the overall distortion and the SNR for a given overall bit rate R of 4 bit per sample.

Determine the high-rate transform coding gain.