**German University in Cairo - GUC**
Faculty of Information Engineering & Technology - IET
Department of Communication Engineering
Winter Semester 2013/2014

**Students Name:** **Students ID:**

# COMM901 – Source Coding and Compression

# Quiz 2

**Note:**

- **The quiz contains 4 questions (each of which has 10 points)**
- **Only the best 3 questions are counted (maximum grade is 30)**

| | Question 1 | Question 2 | Question 3 | Question 4 |
|---|---|---|---|---|
| possible points | 10 | 10 | 10 | 10 |
| achieved point | | | | |
| counted (best 3Q) | | | | |

| | |
|---|---|
| sum counted: | |
| percentage: | |

## Question 1: Mutual Information [10 points]

Given is a stationary Markov process $\boldsymbol{S} = \{S_n\}$ with the binary symbol alphabet $\mathcal{A}_S = \{x, y\}$. The conditional symbol probabilities $p(s_n|s_{n-1}) = P(S_n = s_n|S_{n-1} = s_{n-1})$ are given in the left table below. Due to the symmetry of the transition probabilities, the marginal probability mass function is uniform; the corresponding marginal symbol probabilities $p(s_n) = P(S_n = s_n)$ are given in the right table below.

| $s_n$ | $p(s_n\|s_{n-1} = x)$ | $p(s_n\|s_{n-1} = y)$ |
|---|---|---|
| $x$ | 3/4 | 1/4 |
| $y$ | 1/4 | 3/4 |

| $s_n$ | $p(s_n)$ |
|---|---|
| $x$ | 1/2 |
| $y$ | 1/2 |

Determine the mutual information $I(S_n; S_{n+1})$ between two successive random variables $S_n$ and $S_{n+1}$ of the given stationary Markov source.

The mutual information $I(S_n; S_{n+1})$ between two successive samples is given by

$$I(S_n; S_{n+1}) = H(S_{n+1}) - H(S_{n+1}|S_n) \quad [\mathbf{2}]$$

The marginal entropy $H(S_{n+1})$ is given by

$$H(S_{n+1}) = H(S_n) = -\sum_{\forall i} p(s_i) \cdot \log_2 p(s_i) = -2 \cdot \frac{1}{2} \cdot \log_2 \frac{1}{2} = 1 \quad [\mathbf{2}]$$

For the conditional entropies $H(S_{n+1}|S_n = z)$, with $z$ being $x$ or $y$, we obtain

$$H(S_{n+1}|S_n = z) = -\sum_{\forall i} p(s_i|z) \cdot \log_2 p(s_i|z) = -\frac{3}{4} \cdot \log_2 \frac{3}{4} - \frac{1}{2} \cdot \log_2 \frac{1}{2}$$
$$= \frac{3}{2} - \frac{3}{4} \cdot \log_2 3 + \frac{1}{2} = 2 - \frac{3}{4} \cdot \log_2 3 \approx 0.8113 \quad [\mathbf{2}]$$

Then, the conditional entropy $H(S_{n+1}|S_n)$ is given by

$$H(S_{n+1}|S_n) = \sum_{\forall i} p(a_i) \cdot H(S_{n+1}|S_n = a_i) = H(S_{n+1}|S_n = z) \cdot \sum_{\forall i} p(a_i)$$
$$= H(S_{n+1}|S_n = z) = 2 - \frac{3}{4} \cdot \log_2 3 \approx 0.8113 \quad [\mathbf{2}]$$

Using the derived expression for the marginal entropy $H(S_n)$ and the conditional entropy $H(S_{n+1}|S_n)$, we obtain for the mutual information

$$I(S_n; S_{n+1}) = 1 - 2 + \frac{3}{4} \cdot \log_2 3 = \frac{3}{4} \cdot \log_2 3 - 1 \approx 0.1887 \quad [\mathbf{2}]$$

The mutual information between two successive random variables is 0.1887 bit per sample.

## Question 2: Properties of Rate-Distortion Functions [10 points]

Given are 5 random processes with independent and identically distributed random variables:

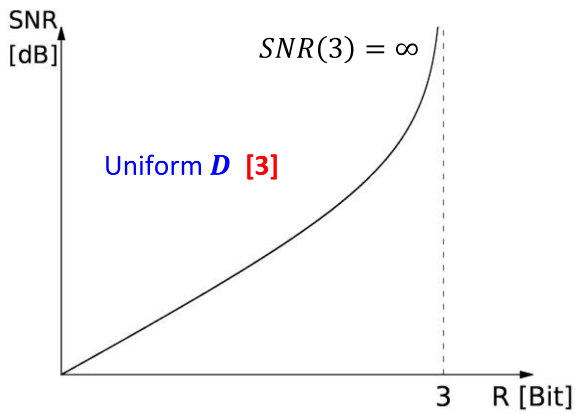
- The random process $\boldsymbol{A}$ is continuous and has a Laplace distribution.
- The random process $\boldsymbol{B}$ is continuous and has a Gaussian distribution.
- The random process $\boldsymbol{C}$ is discrete, has the alphabet $\{x, y\}$ and the pmf $p(x) = p(y) = 1/2$.
- The random process $\boldsymbol{D}$ is discrete, has the 8-symbol alphabet $\mathcal{A}_D = \{a, b, c, d, e, f, g, h\}$ and a uniform pmf on that alphabet (i.e., all symbols have the same probability).
- The random process $\boldsymbol{E}$ is discrete, has the infinite symbol alphabet $\mathcal{A}_E = \{0, 1, 2, 3, \cdots\}$ and the geometric pmf $p(k) = (1/2)^{k+1}$, with $k \in \mathcal{A}_E$.

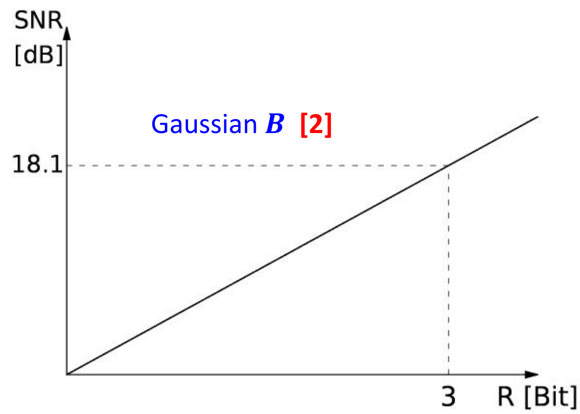
The following diagrams show sketches of the (information) distortion-rate function for four of the five given random processes. As distortion measure the mean squared error (MSE) is used. In the diagrams, the distortion $D$ is represented as signal-to-noise ratio (SNR), given by

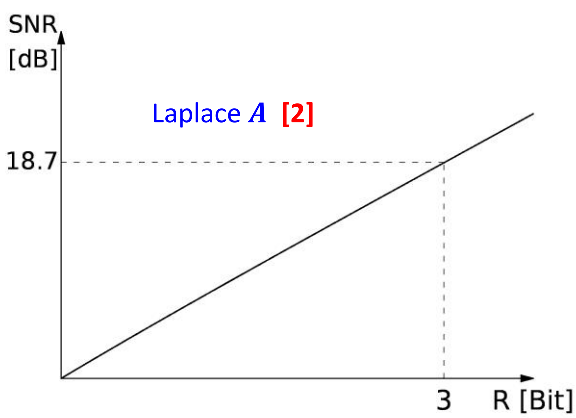
$$\text{SNR} = 10 \cdot \log_{10} \frac{\sigma^2}{D}$$

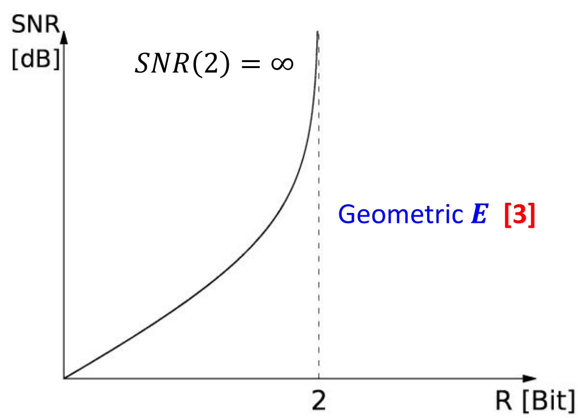
where $\sigma^2$ represents the variance of the random variables.

Assign to each of the diagrams the correct random process ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, *or* $\boldsymbol{E}$).



## Question 3: Usage of Rate-Distortion Functions [10 points]

A student of the German University of Cairo developed a lossy coding algorithm for arbitrary sources. His professor tested the algorithm for a long realization of a stationary Gauss-Markov process and measured a bit rate of $R = 2$ bit per sample and a signal-to-noise ratio of 20 dB.

Given this result, determine the possible values (or intervals) for the correlation coefficient $\varrho$ of the tested Gauss-Markov process.

Hints: *For bit rates greater than 1 bit per sample, the information distortion-rate function for stationary Gauss-Markov processes and MSE distortion is given by*

$$D(R) = (1 - \varrho^2) \cdot \sigma^2 \cdot 2^{-2R}$$

*The signal-to-noise ratio (SNR) is defined by*

$$SNR = 10 \cdot \log_{10} \frac{\sigma^2}{MSE}$$

The distortion-rate function $D(R)$ specifies the minimum distortion that can be achieved by coding a source at a rate $R$. Hence, with $\text{SNR}_{D(R)}$ being the SNR associated with the distortion-rate function and $\text{SNR}_X$ being the SNR of the actual coding result, we have

$$\text{SNR}_X \le \text{SNR}_{D(R)} = 10 \cdot \log_{10} \frac{\sigma^2}{D(R)} = -10 \cdot \log_{10}\left((1 - \varrho^2) \cdot 2^{-2R}\right) \quad [4]$$

By inserting the given values for $R$ and $\text{SNR}_X$, we obtain

$$20 \le -10 \cdot \log_{10}\left((1 - \varrho^2) \cdot 2^{-4}\right)$$

$$-2 \ge \log_{10}\left(\frac{(1 - \varrho^2)}{16}\right)$$

$$\frac{16}{100} \ge 1 - \varrho^2$$

$$\varrho^2 \ge 0.84 \quad [3]$$

This resulting quadratic inequality has the solutions $|\varrho| \ge \sqrt{0.84}$, which is equivalent to

$$\varrho \ge \sqrt{0.84} \approx 0.9165 \qquad \text{and} \qquad \varrho \le -\sqrt{0.84} \approx -0.9165$$

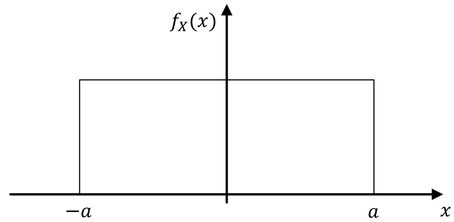
Since a correlation coefficient is always inside the closed interval $[-1; 1]$, it follows that, based on the coding results, the correlation coefficient $\varrho$ lies inside one of the following intervals:

$$-1 \le \varrho \le \sqrt{0.84} \qquad \text{or} \qquad \sqrt{0.84} \le \varrho \le 1 \quad [3]$$

Or in other words, the absolute value of $\varrho$ is bounded by $\sqrt{0.84} \le |\varrho| \le 1$.

## **Question 4: Shannon Lower Bound [10 points]**

Given is a continuous-amplitude random process $\boldsymbol{X} = \{X_n\}$ with independent and identically distributed random variables $X_n$. As illustrated in the figure below, the random variables $X_n$ have a uniform probability density function in an interval $[-a; a]$.




Determine the Shannon lower bound as function of the variance $\sigma^2$. The Shannon lower bound should be formulated as distortion-rate function $D_L(R)$ with the MSE as distortion measure.

Hint: *For MSE distortion, the Shannon lower bound as distortion-rate function is given by*

$$D_L(R) = \frac{1}{2\pi e} \cdot 2^{2\cdot\bar{h}(\boldsymbol{X})} \cdot 2^{-2R}$$

Let $b$ be the value of the pdf inside the interval $[-a; a]$. Since the integral is equal to 1, we have

$$\int_{-a}^{a} f_X(x)\ \mathrm{d}x = b \cdot \int_{-a}^{a} \mathrm{d}x = 2ab = 1 \qquad \Longrightarrow \qquad b = \frac{1}{2a} \quad [2]$$

Since we consider an iid process, the differential entropy rate is equal to the differential entropy:

$$\bar{h}(\boldsymbol{X}) = h(X) = -\int_{-a}^{a} f_X(x) \cdot \log_2 f_X(x)\ \mathrm{d}x = -\int_{-a}^{a} \frac{1}{2a} \cdot \log_2 \frac{1}{2a}\ \mathrm{d}x$$

$$= \frac{\log_2(2a)}{2a} \cdot \int_{-a}^{a} \mathrm{d}x = \frac{\log_2(2a)}{2a} \cdot \big(a - (-a)\big) = \log_2(2a) \quad [3]$$

Hence, we obtain for the Shannon lower bound

$$D_L(R) = \frac{1}{2\pi e} \cdot 2^{2\cdot\bar{h}(\boldsymbol{X})} \cdot 2^{-2R} = \frac{1}{2\pi e} \cdot 2^{2\log_2(2a)} \cdot 2^{-2R} = \frac{4a^2}{2\pi e} \cdot 2^{-2R} = \frac{2a^2}{\pi e} \cdot 2^{-2R} \quad [2]$$

Since we want to formulate the Shannon lower bound as function of the variance $\sigma^2$, we calculate the variance $\sigma^2$. Note that the mean is $\mu = 0$.

$$\sigma^2 = \int_{-a}^{a} x^2 \cdot f_X(x)\ \mathrm{d}x = \frac{1}{2a} \int_{-a}^{a} x^2\ \mathrm{d}x = \frac{1}{2a} \cdot \left( \frac{a^3}{3} - \left( -\frac{a^3}{3} \right) \right) = \frac{a^2}{3} \quad [2]$$

Hence, we have $a^2 = 3\sigma^2$, which yields the Shannon lower bound

$$D_L(R) = \frac{6}{\pi e} \cdot \sigma^2 \cdot 2^{-2R} \quad [1]$$