Exercises with solutions (Set A)

1. Given is a stationary discrete Markov process with the alphabet $\mathcal{A} = \{a_0, a_1, a_2\}$ and the conditional pmfs listed in the table below

a	a_0	a_1	a_2
$p(a\|a_0)$	0.90	0.05	0.05
$p(a\|a_1)$	0.15	0.80	0.05
$p(a\|a_2)$	0.25	0.15	0.60
$p(a)$			

Determine the marginal pmf $p(a)$.

Solution:

Lets consider the probability mass $p(a_0)$. It has to fulfill the following condition:

$$\begin{aligned} p(a_0) &= p(a_0, a_0) + p(a_0, a_1) + p(a_0, a_2) \\ &= p(a_0|a_0) \cdot p(a_0) + p(a_0|a_1) \cdot p(a_1) + p(a_0|a_2) \cdot p(a_2) \\ 0 &= (p(a_0|a_0) - 1) \cdot p(a_0) + p(a_0|a_1) \cdot p(a_1) + p(a_0|a_2) \cdot p(a_2) \end{aligned}$$

Similarly, by considering $p(a_1)$ and $p(a_2)$, we obtain

$$\begin{aligned} 0 &= p(a_1|a_0) \cdot p(a_0) + (p(a_1|a_1) - 1) \cdot p(a_1) + p(a_1|a_2) \cdot p(a_2) \\ 0 &= p(a_2|a_0) \cdot p(a_0) + p(a_2|a_1) \cdot p(a_1) + (p(a_2|a_2) - 1) \cdot p(a_2) \end{aligned}$$

Inserting the given conditional probability masses yields the following linear equation system:

$$\begin{aligned} -0.1 \cdot p(a_0) + 0.15 \cdot p(a_1) + 0.25 \cdot p(a_2) &= 0 \\ 0.05 \cdot p(a_0) - 0.20 \cdot p(a_1) + 0.15 \cdot p(a_2) &= 0 \\ 0.05 \cdot p(a_0) + 0.05 \cdot p(a_1) - 0.40 \cdot p(a_2) &= 0 \end{aligned}$$

After one stage of the Gauss algorithm, we obtain

$$\begin{aligned} -0.25 \cdot p(a_1) + 0.55 \cdot p(a_2) &= 0 \\ 0.25 \cdot p(a_1) - 0.55 \cdot p(a_2) &= 0 \end{aligned}$$

Both equations are linearly dependent, but we obtained the condition

$$0.25 \cdot p(a_1) = 0.55 \cdot p(a_2),$$

which yields

$$p(a_1) = \frac{11}{5} \cdot p(a_2).$$

Using the first equation of the original equation system, we obtain

$$
\begin{aligned}
0.1 \cdot p(a_0) &= 0.15 \cdot p(a_1) + 0.25 \cdot p(a_2) \\
p(a_0) &= \frac{3}{2} \cdot p(a_1) + \frac{5}{2} \cdot p(a_2) \\
p(a_0) &= \frac{3}{2} \cdot \frac{11}{5} \cdot p(a_2) + \frac{5}{2} \cdot p(a_2) \\
p(a_0) &= \frac{33}{10} \cdot p(a_2) + \frac{25}{10} \cdot p(a_2) \\
p(a_0) &= \frac{58}{10} \cdot p(a_2) \\
p(a_0) &= \frac{29}{5} \cdot p(a_2)
\end{aligned}
$$

We know that the pmf has to fulfill the condition

$$p(a_0) + p(a_1) + p(a_2) = 1,$$

which yields

$$
\begin{aligned}
\frac{29}{5} \cdot p(a_2) + \frac{11}{5} \cdot p(a_2) + p(a_2) &= 1 \\
\frac{29 + 11 + 5}{5} \cdot p(a_2) &= 1 \\
\frac{45}{5} \cdot p(a_2) &= 1 \\
p(a_2) &= \frac{1}{9}
\end{aligned}
$$

Inserting into the found expression for $p(a_1)$ and $p(a_0)$ yields

$$
\begin{aligned}
p(a_1) &= \frac{11}{5} \cdot p(a_2) = \frac{11}{5} \cdot \frac{1}{9} = \frac{11}{45} \\
p(a_0) &= \frac{29}{5} \cdot p(a_2) = \frac{29}{5} \cdot \frac{1}{9} = \frac{29}{45}
\end{aligned}
$$

Hence, the marginal pmf is given by

$$
\begin{aligned}
p(a_0) &= \frac{29}{45} = 0.6\bar{4} \\
p(a_1) &= \frac{11}{45} = 0.2\bar{4} \\
p(a_2) &= \frac{1}{9} = 0.\bar{1}
\end{aligned}
$$

2. Investigate the relationship between independence and correlation.

   (a) Two random variables X and Y are said to be *correlated* if and only if their covariance C_{XY} is not equal to 0.

   Can two independent random variables X and Y be correlated?

   <u>*Solution:*</u>

   Without loss of generality, we assume that the statistical properties of the random variables X and Y are given by the joint probability density function $f_{XY}(x,y)$ and marginal probability density functions $f_X(x)$ and $f_Y(y)$. Note that for a discrete random variable X with alphabet $\mathcal{A}$, the pdf $f_X(x)$ can be written using the probability mass function $p_X(a)$ and the Dirac delta function $\delta(x)$,

$$f_X(x) = \sum_{a \in \mathcal{A}} p_X(a) \cdot \delta(x-a).$$

   Similarly, a joint pdf $f_{XY}(x,y)$ can be constructed using the Dirac delta function if either or both random variables X and Y are discrete random variables.

   Two random variables X and Y are independent if and only if the joint pdf is equal to the product of the marginal pdfs, $\forall x, y \in \mathbb{R}$, $f_{XY}(x,y) = f_X(x) f_Y(y)$. For the covariance C_{XY} of two independent random variables X and Y, we then obtain

$$\begin{aligned}
C_{XY} &= E\{(X - E\{X\})\,(Y - E\{Y\})\} \\
&= E\{XY - XE\{Y\} - E\{X\}Y + E\{X\}E\{Y\}\} \\
&= E\{XY\} - E\{X\}E\{Y\} - E\{X\}E\{Y\} + E\{X\}E\{Y\} \\
&= E\{XY\} - E\{X\}E\{Y\} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, y\, f_{XY}(x,y)\, \mathrm{d}x\, \mathrm{d}y - E\{X\}E\{Y\} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, y\, f_X(x)\, f_Y(y)\, \mathrm{d}x\, \mathrm{d}y - E\{X\}E\{Y\} \\
&= \int_{-\infty}^{\infty} x\, f_X(x) \left(\int_{-\infty}^{\infty} y\, f_Y(x)\, \mathrm{d}y \right) \mathrm{d}x - E\{X\}E\{Y\} \\
&= \left(\int_{-\infty}^{\infty} x\, f_X(x)\, \mathrm{d}x \right) \left(\int_{-\infty}^{\infty} y\, f_Y(x)\, \mathrm{d}y \right) - E\{X\}E\{Y\} \\
&= E\{X\}E\{Y\} - E\{X\}E\{Y\} \\
&= 0.
\end{aligned}$$

   Two independent random variables are always uncorrelated.

(b) Let X be a continuous random variable with a variance $\sigma_X^2 > 0$ and a pdf $f_X(x)$. The pdf shall be non-zero for all real numbers, $f_X(x) > 0$, $\forall x \in \mathbb{R}$. Furthermore, the pdf $f_X(x)$ shall be symmetric around zero, $f_X(x) = f_X(-x)$, $\forall x \in \mathbb{R}$. Let Y be a random variable given by $Y = a\,X^2 + b\,X + c$ with $a, b, c \in \mathbb{R}$.
For which values of a, b, and c are X and Y uncorrelated?
For which values of a, b, and c are X and Y independent?

Solution:
First we investigate the correlation of the random variables X and Y. For the covariance C_{XY}, we have

$$\begin{aligned} C_{XY} &= E\{XY\} - E\{X\}E\{Y\} \\ &= E\{aX^3 + bX^2 + cX\} - E\{X\}E\{aX^2 + bX + c\} \\ &= aE\{X^3\} + bE\{X^2\} + cE\{X\} \\ &\quad -aE\{X\}E\{X^2\} - bE\{X\}^2 - cE\{X\} \\ &= a\big(E\{X^3\} - E\{X\}E\{X^2\}\big) + b\big(E\{X^2\} - E\{X\}^2\big) \\ &= a\big(E\{X^3\} - E\{X\}\sigma_X^2\big) + b\big(\sigma_X^2 - E\{X\}^2\big) \end{aligned}$$

Due to the symmetry of the pdf around zero, $f_X(x) = f_X(-x)$, $\forall x \in \mathbb{R}$, the expectation values of the odd integer powers of the random variable X are equal to 0. With the integer variable $n \geq 0$, we have

$$\begin{aligned} E\{X^{2n+1}\} &= \int_{-\infty}^{\infty} x^{2n+1}\, f_X(x)\, \mathrm{d}x \\ &= \int_{0}^{\infty} x^{2n+1}\, f_X(x)\, \mathrm{d}x + \int_{-\infty}^{0} x^{2n+1}\, f_X(x)\, \mathrm{d}x. \end{aligned}$$

Using the substitution $t = -x$ for the second integral, we obtain

$$\begin{aligned} E\{X^{2n+1}\} &= \int_{0}^{\infty} x^{2n+1}\, f_X(x)\, \mathrm{d}x + \int_{\infty}^{0} (-t)^{2n+1}\, f_X(-t)\, (-\mathrm{d}t) \\ &= \int_{0}^{\infty} x^{2n+1}\, f_X(x)\, \mathrm{d}x + \int_{\infty}^{0} t^{2n+1}\, f_X(t)\, \mathrm{d}t \\ &= \int_{0}^{\infty} x^{2n+1}\, f_X(x)\, \mathrm{d}x - \int_{0}^{\infty} t^{2n+1}\, f_X(t)\, \mathrm{d}t \\ &= 0. \end{aligned}$$

In particular, we have $E\{X\} = 0$ and $E\{X^3\} = 0$, yielding

$$\begin{aligned} C_{XY} &= a\big(E\{X^3\} - E\{X\}\sigma_X^2\big) + b\big(\sigma_X^2 - E\{X\}^2\big) \\ &= b \cdot \sigma_X^2. \end{aligned}$$

The random variables X and Y are uncorrelated if and only if b is equal to 0.

Now, we investigate the dependence of the random variables X and Y. The random variables X and Y are independent if and only if $f_{XY}(x, y) = f_X(x)f_Y(y)$. Since $f_{XY}(x, y) = f_{Y|X}(y|x)f_X(x)$, we can also say that X and Y are independent if and only if the marginal pdf for $f_Y(y)$ is equal to the conditional pdf $f_{Y|X}(y|x)$.

The value of the random variable Y is completely determined by the value of the random variable X. Hence, the conditional pdf $f_{Y|X}(y|x)$ is given by the Dirac delta function

$$f_{Y|X}(y|x) = \delta(y - ax^2 - bx - c).$$

If the conditional pdf $f_{Y|X}(y|x)$ depends on the value x of the random variable X, the random variables X and Y are not independent, since $f_Y(y)$ cannot be equal to $f_{Y|X}(y|x)$ in this case. The conditional pdf $f_{Y|X}(y|x)$ does not depend on x if one of the following conditions is fulfilled:

- $a \neq 0$ and $f_X(x) = \frac{w}{2}\delta(x - x_1) + \frac{1-w}{2}\delta(x - x_2)$, where x_1 and x_2 are the roots of the quadratic equation $ax^2 + bx = d$ for any value of $d > -b^2/(4a)$, and $0 \leq w \leq 1$;
- $a = 0$, $b \neq 0$, and $f_X(x) = \delta(x - x_0)$ with x_0 being any constant real value;
- $a = 0$ and $b = 0$.

Since it is given that $f_X(x) > 0$, $\forall x \in \mathbb{R}$, we do not need to consider the first two cases. Hence, for all parameters a, b, and c with $a \neq 0$ or $b \neq 0$, the random variables X and Y are dependent.

For the case $a = 0$ and $b = 0$, the conditional pdf is given by

$$f_{Y|X}(y|x) = \delta(y - c),$$

and the random variable Y is given by $Y = c$. The random variable Y is always equal to c. Consequently, its marginal pdf is given by

$$f_Y(y) = \delta(y - c)$$

and is equal to the conditional pdf $f_{Y|X}(y|x)$.

The random variables X and Y are independent if and only if $a = 0$ and $b = 0$.

(c) Which of the following statements for two random variables X and Y are true?

i. If X and Y are uncorrelated, they are also independent.
ii. If X and Y are independent, $E\{XY\} = 0$.
iii. If X and Y are correlated, they are also dependent.

<u>*Solution:*</u>

i. The statement "if X and Y are uncorrelated, they are also independent" is wrong. As a counterexample consider the random variables X and Y in problem (2b) for $a \neq 0$ and $b = 0$. In this case, the random variables are uncorrelated, but are dependent.

ii. The statement "if X and Y are independent, $E\{XY\} = 0$" is wrong. As shown in problem (2a), the independence of X and Y implies $C_{XY} = E\{XY\} - E\{X\}E\{Y\} = 0$. If, however, both mean values $E\{X\}$ and $E\{Y\}$ are not equal to zero, then $E\{XY\}$ is also not equal to zero.

iii. The statement "if X and Y are correlated, they are also dependent" is true. This statement is the contraposition of the statement "if X and Y are independent, they are also uncorrelated", which has been proved in problem (2a).