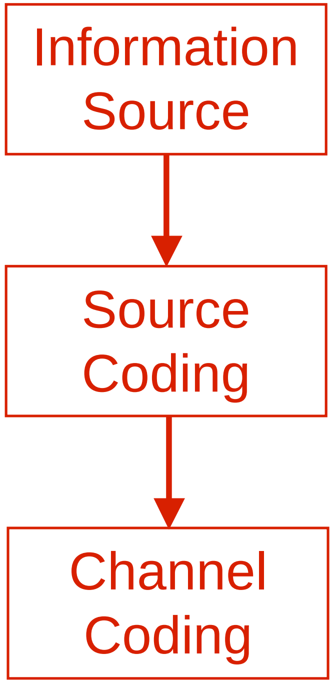
Information and Entropy

- Shannon’s Separation Principle
- Source Coding Principles
- Entropy
- Variable Length Codes
- Huffman Codes
- Joint Sources
- Arithmetic Codes
- Adaptive Codes

Shannon's Separation Principle

Assumptions:

- Single source and user
- Unlimited complexity and delay



Information Source: Generates information we want to transmit or store

Source Coding: Reduces number of bits to store or transmit relevant information

Channel Coding: Increases number of bits or changes them to protect against channel errors

Practical Systems

- Many applications are not uni-directional point-to-point transmissions:
  - Feedback
  - Networks

- In any practical system, we cannot effort unlimited complexity *neither unlimited delay:*
  - There will always be a small error rate unless we tolerate sub-optimality
  - It might work better to consider source and channel coding jointly
  - Consider effect of transmission errors on source decoding result

Source Coding Principles

- The source coder shall represent the video signal by the minimum number of (binary) symbols without exceeding an acceptable level of distortion.
- Two principles are utilized:

1. Properties of the information source that are known a priori result in redundant information that need not be transmitted (“redundancy reduction“).

2. The human observer does not perceive certain deviations of the received signal from the original (“irrelevancy reduction“).

- *Lossless* coding: completely reversible, exploit 1. principle only
- *Lossy* coding: not reversible, exploit 1. and 2. principle

Entropy of a Memoryless Source

- Let a memoryless source be characterized by an ensemble U_0 with:

Alphabet $\{ a_0, a_1, a_2, \ldots, a_{K-1} \}$

Probabilities $\{ P(a_0), P(a_1), P(a_2), \ldots, P(a_{K-1}) \}$

- Shannon: information conveyed by message “a_k“:

$$I(a_k) = -\log(P(a_k))$$

- “Entropy of the source“ is the average information contents:

$$H(U_0) = E\{I(a_k)\} = -\sum_{k=0}^{K-1} P(a_k) * \log(P(a_k))$$

- For „log“ = „$\log_2$“ the unit is bits/symbol

Entropy and Bit-Rate

- Properties of entropy**:**

$$H(U_0) \geq 0$$

$$\max \{ H(U_0) \} = \log K \quad \text{with} \quad P(a_j) = P(a_k) \text{ for all } j, k$$

- The entropy $H(U_0)$ is a lower bound for the average word length λ_{av} of a decodable variable length code with $\lambda_{cw}(a_k)$ being individual code word lengths

$$\lambda_{av} = \sum_{k=0}^{K-1} P(a_k) * \lambda_{cw}(a_k)$$

- Conversely, the average word length λ_{av} can approach $H(U_0)$, if sufficiently large blocks of symbols are encoded jointly.

- Redundancy of a code: $\rho = \lambda_{av} - H(U_0) \geq 0$

Encoding with Variable Word Lengths

- A code without redundancy, i.e.

$$\boxed{\lambda_{av} = H(U_0)}$$

is achieved, if all individual code word lengths

$$\boxed{\lambda_{cw}(a_k) = -\log(P(a_k))}$$

- For binary code words, all probabilities would have to be binary fractions:

$$\boxed{P(a_k) = 2^{-\lambda_{cw}(a_k)}}$$

Redundant Codes: Example

a_i	$P(a_i)$	redundant code	optimum code
a_1	0.500	00	0
a_2	0.250	01	10
a_3	0.125	10	110
a_4	0.125	11	111

$H(U_0)$ =1.75 bits

λ_{av}= 2 bits
ρ = 0.25 bits

λ_{av}=1.75 bits
ρ = 0 bits

Variable Length Codes

- Unique decodability: Where does each code word start or end
- Insert start symbol: 01.0.010.1. wasteful
- Construct prefix-free code
- Kraft Inequality: test for uniquely decodable codes

Uniquely decodable code exists if $\varsigma = \sum_{k=0}^{K-1} 2^{-\lambda_{cw}(a_k)} \leq 1$

- Application:

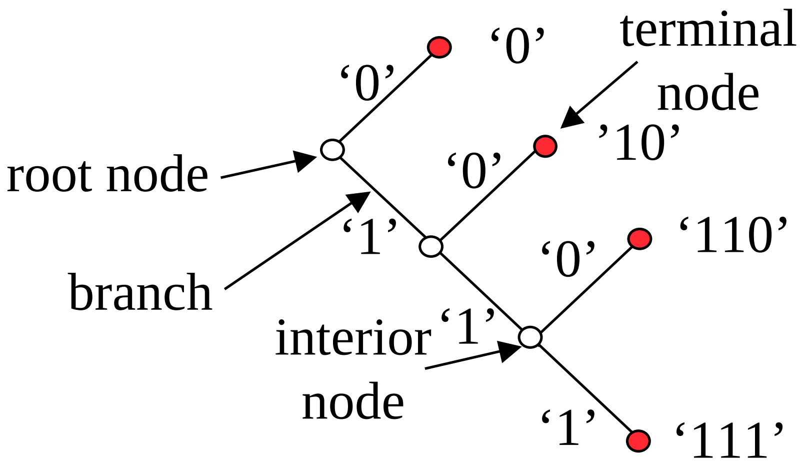
a_i	$P(a_i)$	$-\log_2(P(a_i))$	Code A	Code B
a_1	0.5	1	0	0
a_2	0.2	2.32	01	10
a_3	0.2	2.32	10	110
a_4	0.1	3.32	111	111
			$\varsigma = 1.125$	$\varsigma = 1$

Not uniquely decodable

Uniquely decodable

Prefix-Free Codes

- Prefix-free codes are instantaneously and uniquely decodable
- Prefix-free codes can be represented by trees



- Terminal nodes may be assigned code words
- Interior nodes cannot be assigned code words
- For binary trees: *N* terminal nodes: *N-1* interior nodes

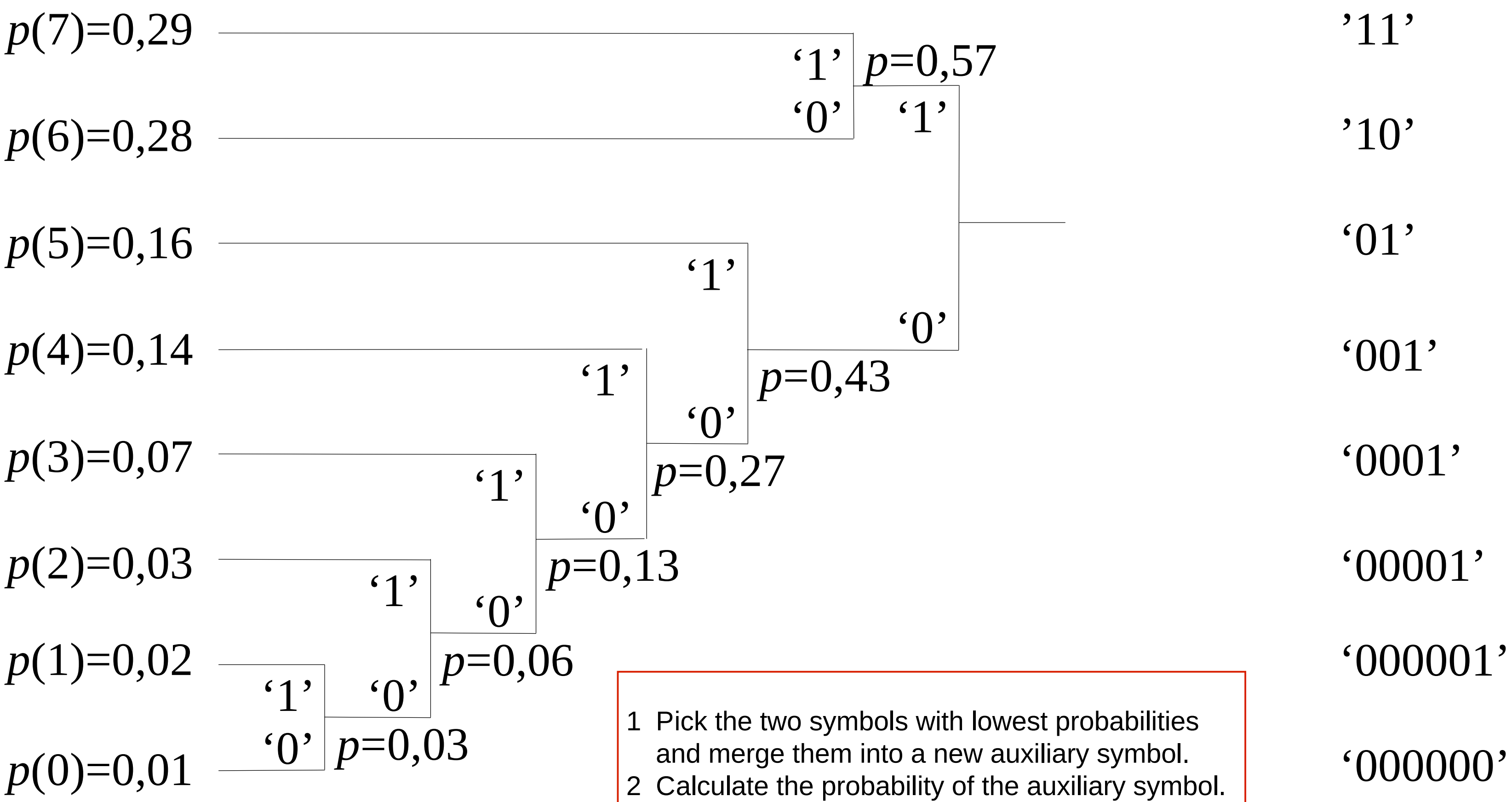
- Code 0, 01, 11 is not a prefix-free code and uniquely decodable but: non-instantaneous

Huffman Code

- Design algorithm for variable length codes proposed by D. A. Huffman (1952) always finds a code with minimum redundancy.
- Obtain code tree as follows:

1. Pick the two symbols with lowest probabilities and merge them into a new auxiliary symbol.
2. Calculate the probability of the auxiliary symbol.
3. If more than one symbol remains, repeat steps 1 and 2 for the new auxiliary alphabet.
4. Convert the code tree into a prefix code.

Huffman Code: Example



$p(7)=0{,}29$ ’11’

$p(6)=0{,}28$ ’10’

$p(5)=0{,}16$ ‘01’

$p(4)=0{,}14$ ‘001’

$p(3)=0{,}07$ ‘0001’

$p(2)=0{,}03$ ‘00001’

$p(1)=0{,}02$ ‘000001’

$p(0)=0{,}01$ ‘000000’

1. Pick the two symbols with lowest probabilities and merge them into a new auxiliary symbol.
2. Calculate the probability of the auxiliary symbol.
3. If more than one symbol remains, repeat steps 1 and 2 for the new auxiliary alphabet.
4. Convert the code tree into a prefix code.

Joint Sources

- Joint sources generate *N* symbols simultaneously.
- A coding gain can be achieved by encoding those symbols jointly.
- The lower bound for the average code word length is the joint entropy:

$$\boxed{H(U_1, U_2, \cdots, U_N) = -\sum_{u_1}\sum_{u_2}\cdots\sum_{u_N} P(u_1, u_2, \cdots, u_N)\cdot\log\left(P(u_1, u_2, \cdots, u_N)\right)}$$

- It generally holds that

$$\boxed{H(U_1, U_2, \cdots, U_N) \leq H(U_1) + H(U_2) + \cdots + H(U_N)}$$

with equality, if U_1, U_2, ..., U_N are statistically independent.

Markov Process

- Neighboring samples of the video signal are **not** statistically independent:

Source with memory

$$\boxed{P(u_T) \neq P(u_T \mid u_{T-1}, u_{T-2}, \ldots, u_{T-N})}$$

- A source with memory can be modeled by a Markov random process.
- Conditional probabilities of the source symbols u_T of a Markov source of order N:

$$\boxed{P(u_T \mid Z_T) = P(u_T \mid u_{T-1}, u_{T-2}, \ldots, u_{T-N})}$$

state of the Markov source at time T

Entropy of Source with Memory

- Markov source of order *N*: conditional entropy

$$
\begin{aligned}
H(U_T \mid Z_T) &= H(U_T \mid U_{T-1}, U_{T-2}, \ldots, U_{T-N}) \\
&= -E\{\log(p(u_T \mid u_{T-1}, u_{T-2}, \ldots, u_{T-N}))\} \\
&= -\sum_{u_T} \ldots \sum_{u_{T-N}} p(u_T, u_{T-1}, u_{T-2}, \ldots, u_{T-N}) \log(p(u_T \mid u_{T-1}, u_{T-2}, \ldots, u_{T-N}))
\end{aligned}
$$

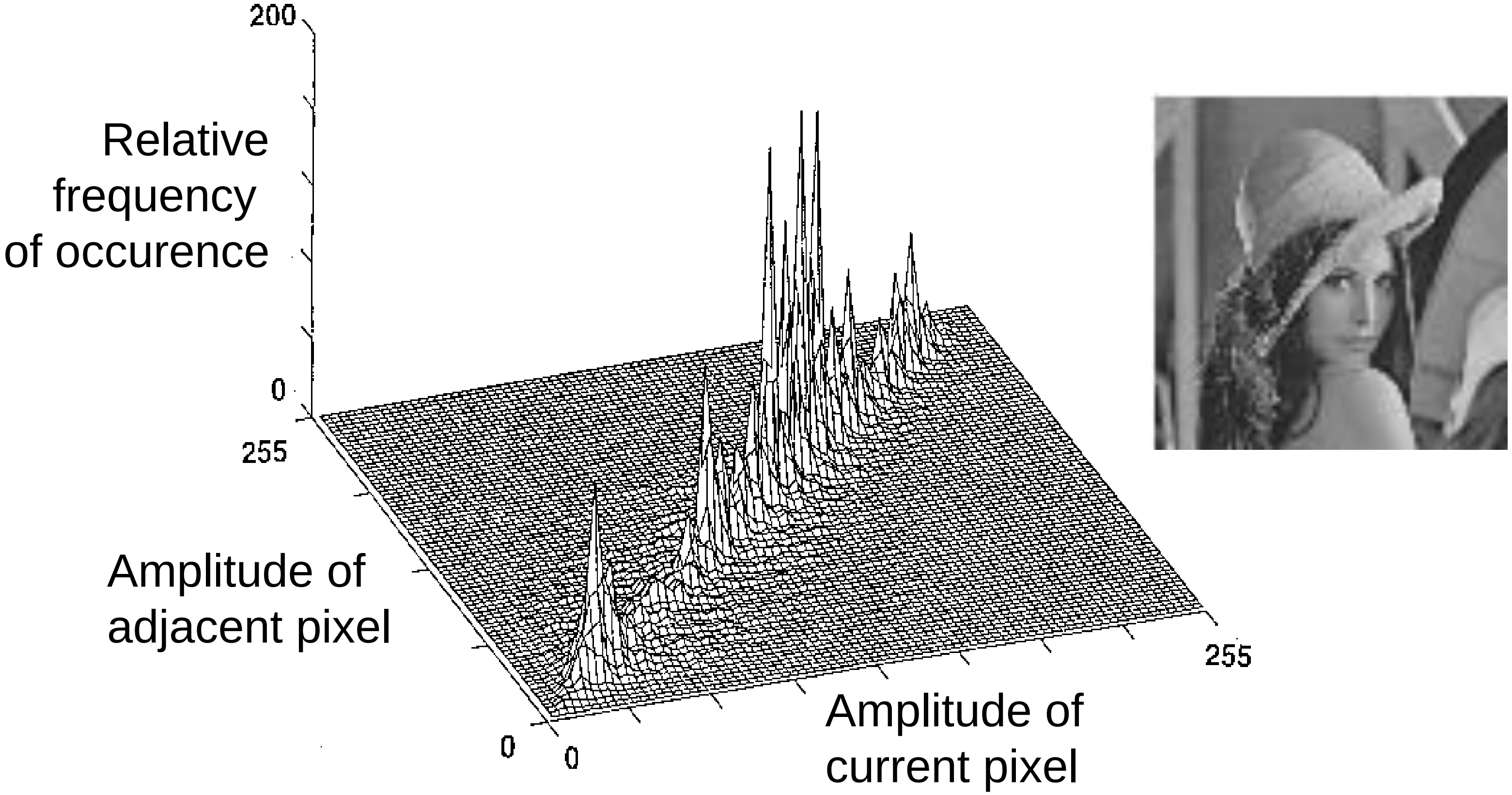
$$H(U_T) \geq H(U_T \mid Z_T)$$

(equality for memoryless source)

- Average code word length can approach $H(U_T \mid Z_T)$ e.g. with a switched Huffman code
- Number of states for an 8-bit video signal:

$N = 1$ → 256 states
$N = 2$ → 65536 states
$N = 3$ → 16777216 states

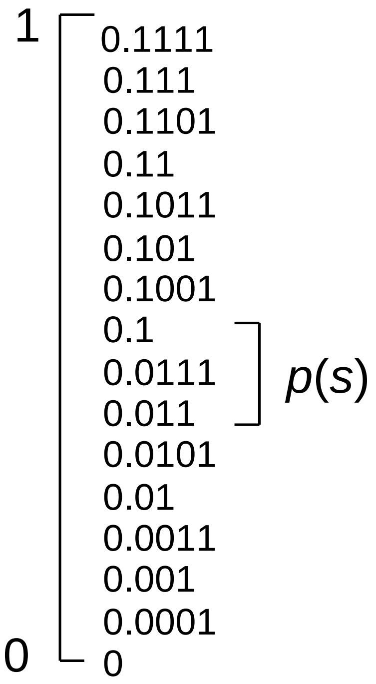
Second Order Statistics for Luminance Signal



Histogram of two horizontally adjacent pels
(picture: female head-and-shoulder view)

Arithmetic Coding

- Universal entropy coding algorithm for strings
- Representation of a string by a subinterval of the unit interval [0,1)
- Width of the subinterval is approximately equal to the probability of the string $p(s)$



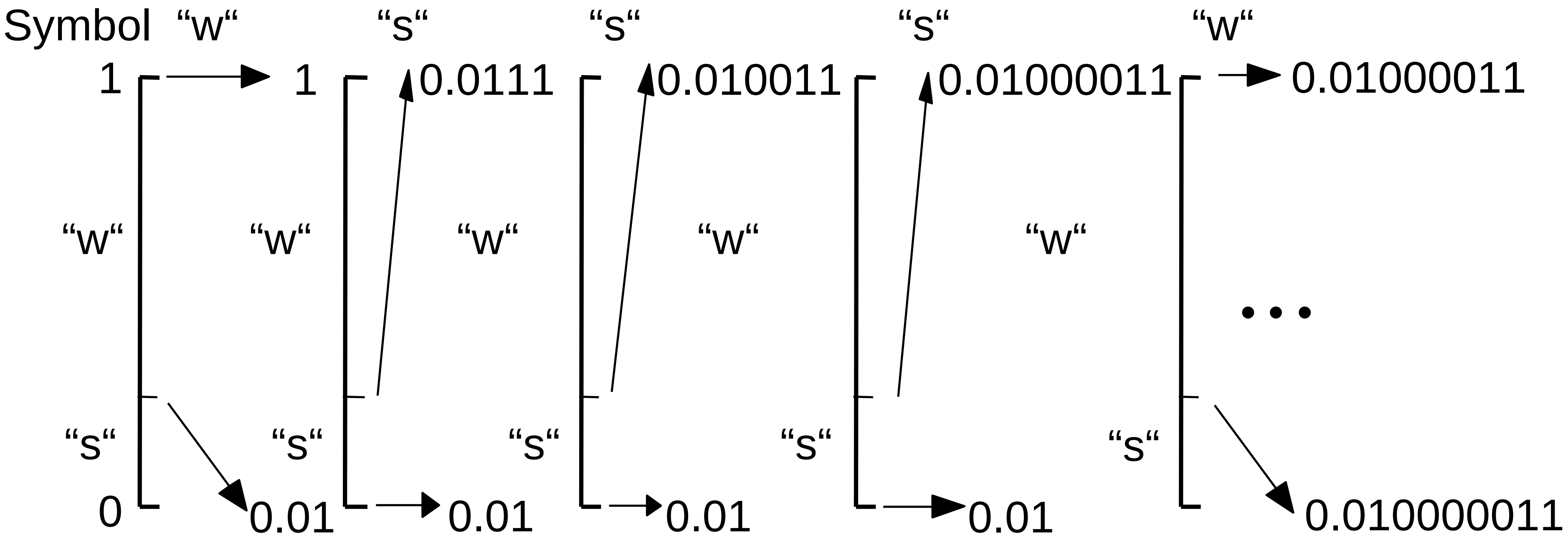
- Interval of width $p(s)$ is guaranteed to contain one number that can be represented by b binary digits, with

$$-\log(p(s)) + 1 \leq b \leq -\log(p(s)) + 2$$

- Each interval can be represented by a number which needs 1 to 2 bits more than the ideal code word length

Arithmetic Coding: Probability Intervals

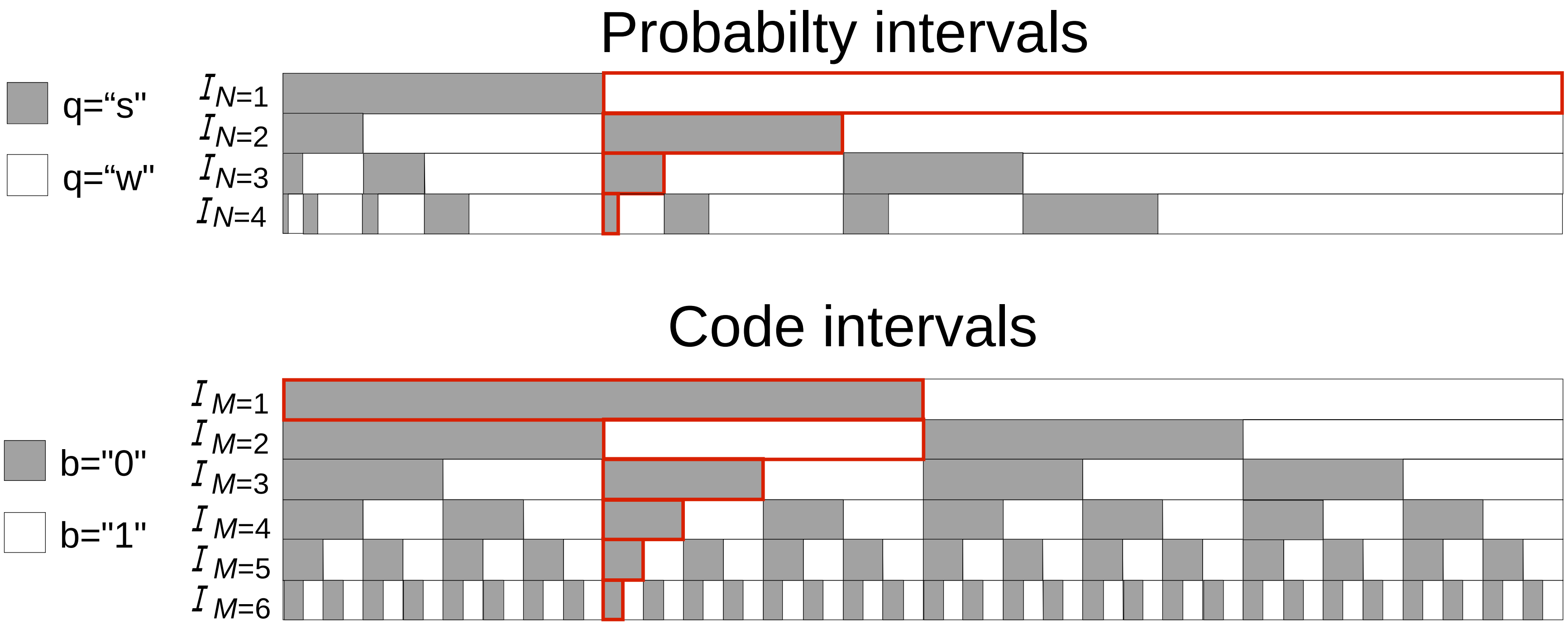
- Random experiment: pmf $p(\text{“s“}) = (0.01)_b$ and $p(\text{“w“}) = (0.11)_b$



- Multiplications and additions with (potentially) very long word length
- Universal coding: probabilities can be changed on the fly:
  e.g., use $p(\text{“s“} \mid \text{“s“})$, $p(\text{“s“} \mid \text{“w“})$, $p(\text{“w“} \mid \text{“s“})$, $p(\text{“w“} \mid \text{“w“})$

Arithmetic Encoding and Decoding

- Encoding: “w”, “s”, “s”, “s” ➔ 010000
- Decoding: 010 ➔ “w”, “s”

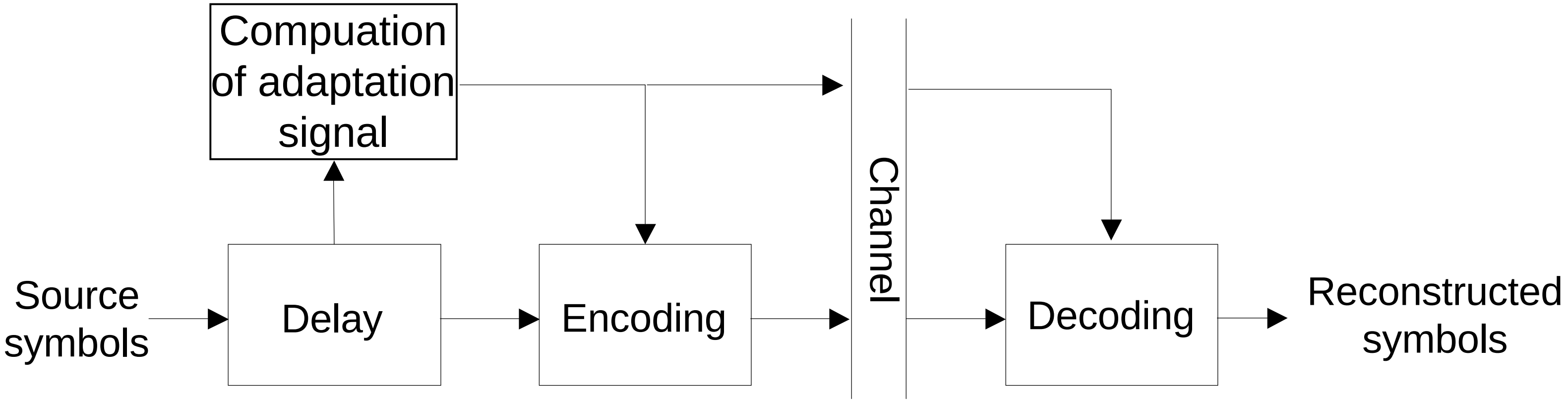


Adaptive Entropy Coding

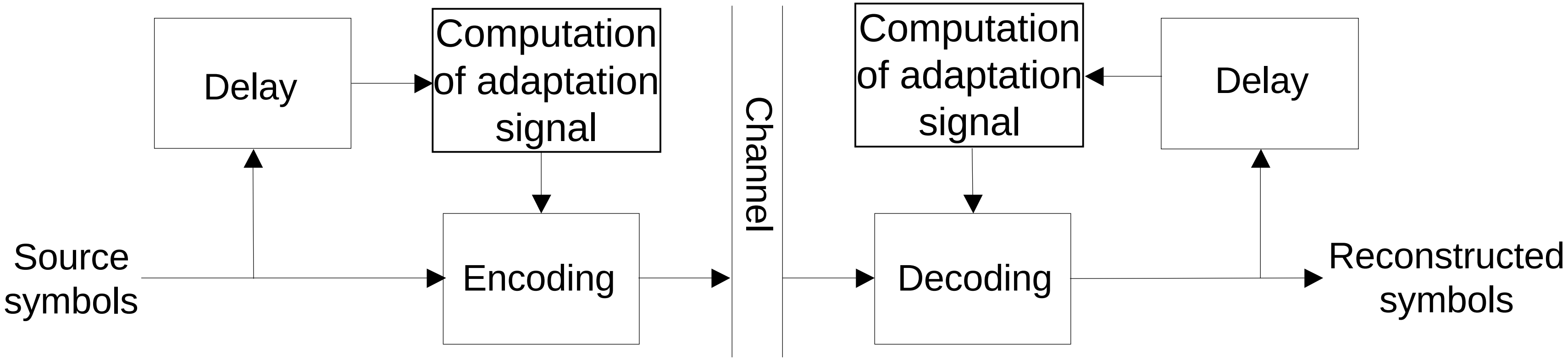
- For non-adaptive coding methods: pdf of source must be known a priori (inherent assumption: stationary source)
- Image and video signals are not stationary: sub-optimal performance
- Solution: adaptive entropy coding
- Two basic approaches to adaptation:
  1. Forward Adaptation
     - Gather statistics for a large enough block of source symbols
     - Transmit adaptation signal to decoder as side information
     - Drawback: increased bit-rate
  2. Backward Adaptation
     - Gather statistics simultaneously at coder and decoder
     - Drawback: error resilience
- Combine the two approaches and circumvent drawbacks (Packet based transmission systems)

Forward vs. Backward Adaptive Systems

Forward Adaptation



Backward Adaptation



Summary

- Shannon’s information theory vs. practical systems
- Source coding principles: redundancy & irrelevancy reduction
- Lossless vs. lossy coding
- Redundancy reduction exploits the properties of the signal source.
- Entropy is the lower bound for the average code word length.
- Huffman code is optimum entropy code.
- Huffman coding: needs code table.
- Arithmetic coding is a universal method for encoding strings of symbols.
- Arithmetic coding does not need a code table.
- Adaptive entropy coding: gains for sources that are not stationary